Boundary Conditions on the Monopole Dirac Equation

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Boundary conditions on the monopole Dirac equation are, on the one hand, determined by the requirement that the Hamiltonian be Hermitean. On the other, they enforce the conservation of certain charges at the monopole core. Which charges should be conserved is determined by the short-distance physics of the problem and the connection with boundary conditions is worked out for several cases. It is found to be possible to impose physically reasonable conservation-law requirements which can be realized by no Dirac-equation boundary condition!

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The peculiarities of fermion-monopole interactions in the S-wave sector give rise to interesting physical effects including catalysis of proton decay.[1,2] In studies of these phenomena the details of the monopole core are not very important: It suffices to consider a point singular monopole and impose boundary conditions on the fermions which summarize the effect of the monopole core. In this Letter we shall examine the role played by these boundary conditions and study the extent to which they are determined by requirements of conservation of certain charges at the monopole center. We shall discover that in the physically interesting case of more than one flavor of fermion, there is an incompatibility between conservation laws and the standard picture of Dirac-equation boundary conditions. This incompatibility has much to do with the peculiar physics of monopole catalysis of proton delay.

We consider several flavors of Dirac fermions having charges $\pm e$ in the field of an Abelian point monopole with a magnetic charge $g = 1/2e$. This monopole can be thought of as the exterior field of a unified-gauge-theory monopole whose core is so small that it can for most purposes be thought of as a point. Each pair of fermions having charges $+e$ and $-e$ can be thought of as a single $I = \frac{1}{2}$ multiplet. In the following a single flavor refers to one such multiplet.

The Dirac wave function has a well-known decomposition in terms of eigenstates of the total angular momentum operator[3]

$$\vec{J} = \vec{L} + \vec{S} + q\hat{r},$$

where $\vec{L}$ and $\vec{S}$ denote the orbital and spin angular momenta and $q \equiv eg$. For familiar reasons, we will concentrate on the $J=0$ partial wave. The $J=0$ wave function may be written in the standard form:

$$\psi_{J=0} = \frac{1}{r}\begin{pmatrix} u(r)\eta_0 \\ v(r)\eta_0 \end{pmatrix}, \tag{1}$$

where η_0 is a two-component monopole harmonic satisfying

$$(\vec{\sigma}\cdot\hat{r})\eta_0 = (q/|q|)\eta_0. \tag{2}$$

In terms of the two-component radial wave function

$$\chi(r) \equiv \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} \tag{3}$$

the Dirac equation becomes

$$(i\gamma_5 q\, d/dr - \gamma_0 m)\chi = H\chi, \tag{4}$$

where

$$\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is convenient to express the wave functions in terms of eigenstates of γ_5:

$$\gamma_5\chi_L = -\chi_L, \quad \gamma_5\chi_R = \chi_R.$$

When $m=0$, the equations for χ_L and χ_R decouple as usual. The one-component wave functions $\chi_{R\pm}(r)$ and $\chi_{L\pm}(r)$ are now labeled by eigenvalues of γ_5 (L or R) and of q (+ or −). In wave-function labeling, the quantity $\epsilon = q\gamma_5$ plays a special role. The eigenvalues of ϵ are ± 1 and the corresponding wave functions are denoted as $\chi_\pm(r)$. Particles and antiparticles have the same value of ϵ and the number of states with $\epsilon = +$ and $\epsilon = -$ are always equal. Furthermore, the ϵ eigenstates have the crucial property that for positive energy $\epsilon = +$ states are always outgoing waves while ϵ

$=-$ states are always incoming waves.

The S-wave Hamiltonian in Eq. (4) is not self-adjoint. However, there exist self-adjoint extensions[4] obtained by restricting the space of states on which it acts. Consider N Dirac particles in the field of a monopole (i.e., $N/2$ flavors). The self-adjointness condition

$$(\tilde{\psi}, H\psi) = (H\tilde{\psi}, \psi),$$

for arbitrary $\tilde{\psi}$ and ψ, becomes, in terms of the ϵ eigenstates,

$$\sum_{i=1}^{N} \tilde{\chi}_{i+}{}^{*}\chi_{i+}\Big|_{r=0} = \sum_{i=1}^{N} \tilde{\chi}_{i-}{}^{*}\chi_{i-}\Big|_{r=0}. \tag{5}$$

The obvious restriction on the space of χ's which achieves this is

$$\chi_{i+}(0) = \sum_{j=1}^{N} U_{ij}\chi_{j-}(0), \tag{6}$$

where U_{ij} is an *arbitrary* $N\times N$ unitary matrix. Note that one has $2N$ one-component fields and that N *linear* conditions (i.e., one condition per Dirac fermion) are required to define the one-particle Dirac equation.

The main pecularity of the $J=0$ partial wave is that various charges may "leak" into the monopole core at $r=0$. This is because the flux of a charge $\hat{Q}$, given by $4\pi r^2 J_r^{\hat{Q}}$ (where $J_r^{\hat{Q}}$ is the radial component of the corresponding current density), need not vanish at $r=0$. It is possible to ensure conservation of some charges by properly specifying the matrix U in the boundary condition (6). One might think that U should be *determined* by requiring that conserved charges should not leak into $r=0$.[5] We shall see shortly that this is not always the case. Consider, for example, a single doublet of Dirac fermions. An expression for the radial fluxes of fermion number, charge, and axial charge, respectively, can quickly be written down by remembering that χ_{R+} and χ_{L-} (χ_{L+} and χ_{R-}) are outgoing (incoming) waves:

$$\begin{aligned}4\pi r^2(\bar{\psi}\gamma_r\psi) &= F_r = |\chi_{R+}|^2 - |\chi_{L+}|^2 - |\chi_{R-}|^2 + |\chi_{L-}|^2,\\ 4\pi r^2(\psi\gamma_r\tau_3\psi) &= J_r\\ &= |\chi_{R+}|^2 - |\chi_{L+}|^2 + |\chi_{R-}|^2 - |\chi_{L-}|^2,\\ 4\pi r^2(\bar{\psi}\gamma_r\gamma_5\psi) &\\ &= J_r^5 = |\chi_{R+}|^2 + |\chi_{L+}|^2 - |\chi_{R-}|^2 - |\chi_{L-}|^2.\end{aligned} \tag{7}$$

If, for example, $J_r(0)\neq 0$, electric charge leaks into the monopole core. To guarantee charge conservation we must impose a *quadratic* condition on the fields at $r=0$ rather than the linear condition required by Hermiticity. In the following we shall examine some particular cases to determine whether it is possible to implement these quadratic relations as *linear* conditions on the Dirac equation, thus determining the boundary-condition matrix U.

Case I: Single Dirac particle of charge +e.—The two one-component fields are χ_R and χ_L. The most general linear boundary condition is

$$\chi_R(0) = e^{i\alpha}\chi_L(0), \tag{8}$$

α being arbitrary. The two relevant charge fluxes are

$$F_r = J_r = |\chi_R|^2 - |\chi_L|^2, \quad J_r^5 = |\chi_R|^2 + |\chi_L|^2. \tag{9}$$

Obviously, it is impossible to have $J_r^5(0)=0$, so that axial charge is never conserved. However, the boundary condition (8) automatically ensures that fermion number and electric charge are conserved for any α. Thus the set of charge-flux conservation laws has the same content as the Hermiticity condition.

Case II: A single flavor (i.e., one doublet).—There are now four one-component fields: $\chi_{R\pm}$ and $\chi_{L\pm}$. The boundary conditions (6) take the form

$$\begin{pmatrix}\chi_{R+}(0)\\ \chi_{L-}(0)\end{pmatrix} = U\begin{pmatrix}\chi_{R-}(0)\\ \chi_{L+}(0)\end{pmatrix}, \tag{10}$$

where U is a 2×2 unitary matrix. There are now *two* linear boundary conditions, and one might try to produce the same effect by imposing two independent conservation-law conditions. We shall explore two possibilities:

(a) $F_r(0)=J_r^5(0)=0$ (conservation of fermion number and axial charge). From the flux expressions in Eq. (7) this implies

$$|\chi_{R+}(0)|^2 = |\chi_{R-}(0)|^2, \quad |\chi_{L+}(0)|^2 = |\chi_{L-}(0)|^2. \tag{11}$$

These conditions can be easily seen to be realizable by

$$U = \begin{pmatrix} e^{i\alpha} & 0\\ 0 & e^{i\beta}\end{pmatrix}. \tag{12}$$

This choice of U does not conserve electric charge.

(b) $F_r(0)=J_r(0)=0$ (conservation of fermion number and electric charge). This is realized by

$$|\chi_{R+}(0)|^2 = |\chi_{L+}(0)|^2, \quad |\chi_{R-}(0)|^2 = |\chi_{L-}(0)|^2, \tag{13}$$

which is equivalent to the Hermiticity boundary conditions with

$$U = \begin{pmatrix} 0 & e^{i\alpha}\\ e^{i\beta} & 0\end{pmatrix}. \tag{14}$$

Therefore in this case a special choice of linear boundary conditions is equivalent to a set of quadratic conditions dictated by conservation laws.

Case III: Two $I=\frac{1}{2}$ multiplets χ and ω.—This is the relevant fermion content in the context of a monopole in a grand unified theory.[1,2] One now has eight single-component fields: $\chi_{R\pm}$, $\chi_{L\pm}$, $\omega_{R\pm}$, and $\omega_{L\pm}$, and the boundary conditions in Eq. (6) translate into four linear relations:

$$\begin{pmatrix}\chi_{R+}\\ \chi_{L-}\\ \omega_{R+}\\ \omega_{L-}\end{pmatrix}=U\begin{pmatrix}\chi_{L+}\\ \chi_{R-}\\ \omega_{L+}\\ \omega_{R-}\end{pmatrix}. \tag{15}$$

The underlying field theory would require four charges to be rigorously conserved: the electric charge, the fermion number of each multiplet, and the *difference* of the axial charges of the two multiplets. The *sum* of the axial charges is not conserved because of an anomaly, but the anomaly cancels in the difference.

The corresponding fluxes may be read off from a trivial extension of Eqs. (7). Thus,

$$F_r^{\chi}(0)=0$$

implies

$$|\chi_{R+}(0)|^2+|\chi_{L-}(0)|^2=|\chi_{L+}(0)|^2+|\chi_{R-}(0)|^2, \tag{16a}$$

while $F_r^{\omega}(0)=0$ implies

$$|\omega_{R+}(0)|^2+|\omega_{L-}(0)|^2=|\omega_{L+}(0)|^2+|\omega_{R-}(0)|^2. \tag{16b}$$

These conditions are realizable in the set of linear conditions of Eq. (15) if U is of the block-diagonal form:

$$U=\begin{pmatrix}u_1 & 0\\ 0 & u_2\end{pmatrix}, \tag{17}$$

where u_1 and u_2 are 2×2 matrices. Conservation of the net electric charge requires

$$J_r^{\chi}(0)+J_r^{\omega}(0)=0,$$

which means

$$|\chi_{R+}(0)|^2-|\chi_{L-}(0)|^2+|\omega_{R+}(0)|^2-|\omega_{L-}(0)|^2=|\chi_{L+}(0)|^2-|\chi_{R-}(0)|^2+|\omega_{L+}(0)|^2-|\omega_{R-}(0)|^2. \tag{18}$$

Conservation of the difference of axial charges requires

$$J_r^{5\chi}(0)-J_r^{5\omega}(0)=0,$$

which becomes

$$|\chi_{R+}(0)|^2-|\chi_{L-}(0)|^2-|\omega_{R+}(0)|^2+|\omega_{L-}(0)|^2=-|\chi_{L+}(0)|^2+|\chi_{R-}(0)|^2+|\omega_{L+}(0)|^2-|\omega_{R-}(0)|^2. \tag{19}$$

It is straightforward to see that the two conditions (18) and (19) cannot be satisfied for any choice of u_1 and u_2. The incompatibility of conservation-law boundary conditions with the Hermiticity conditions is in fact a generic feature for situations in which there are two or more fermion flavors. This does not mean that the conservation laws are inconsistent, but only that they cannot be realized on a single-particle–state space described by the Dirac equation with linear boundary conditions.

Our discussion drives home a major lesson: With a large enough complement of fields, some physically reasonable sets of conservation-law boundary conditions cannot be realized as linear conditions on the Dirac equation. This does not mean that the conservation laws are inconsistent: Rather they imply nontrivial particle production and are not realizable on the single-particle–state space of the field theory. This in fact provides an explanation of why phenomena like monopole catalysis of proton decay occur only if we have two or more Dirac doublets. It also indicates that calculation of the catalysis S matrix is a complicated dynamical problem.

This problem can be seen in a slightly different light by remembering that, through the device of bosonization,[2] the s-wave fermion-monopole system can be rewritten as an equivalent scalar-field theory. Although the scalar fields are complicated nonlinear functions of the original Fermi fields, all of the currents we have considered turn out to be linear functions of the scalar fields. For that reason, any consistent set of conservation-law boundary conditions reduces to a set of linear conditions on the scalar fields, no matter how many Fermi multiplets there originally were. Since partial differential equations with linear boundary conditions are easy to think about, the scalar language is quite useful for qualitative insight even though it does not solve the problem of calculating the S matrix.

On the other hand, since the scalar field is a more or less explicit function of the Fermi fields, one might hope to convert the simple linear scalar-field boundary condition into a complicated, nonlinear but explicit, condition on the Fermi

Fock space from which to read off the S matrix. At the moment, this is but a vague idea on our part, but we hope one of our readers will see how to realize it.

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[5]For more details on the connection between conservation laws and boundary conditions, see A. Sen, Fermilab Report No. FERMILAB-PUB-83/28-THY, 1983 (to be published); C. Callan, "The Monopole Catalysis S Matrix" (to be published).