Extension of the Soft-Photon Theorem for Bremsstrahlung

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A correction to the soft-photon theorem for bremsstrahlung is derived for problems in which the strong interaction is known at large distances. In studies of a model problem the correction generally results in significant improvement in accuracy over the original theorem.

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In a number of studies[1] of the sensitivity of nucleon-nucleon bremsstrahlung to variation in the potential, it has been found that if these changes are confined to sufficiently small distances, *and they preserve the phase shifts*, then the bremsstrahlung cross section is not strongly affected. These calculations were motivated by the belief that beyond some distance (approximately 1 fm) the nucleon-nucleon interaction is describable by a known potential, due to the exchange of one and two π mesons and one ω meson. At small distances, however, where the structure of the nucleon plays a vital role, the nature of the interaction is not well established.

This situation prompted Heller and Low[2] to formulate an extension of the soft-photon theorem for bremsstrahlung.[3] The idea is simple. If the interaction really is known beyond some distance, then the contribution to the bremsstrahlung amplitude from that much of the interaction ought to be calculated exactly. The contribution from the (short-range) remainder is treated in precisely the same way that the full interaction was in the original derivation. Although a proof of the extended theorem appears in Ref. 2, a simpler presentation is given here.

We first give a brief summary of the original version[3] of the theorem. It says that one can compute the amplitude for the process

$$A + B \to C + D + \text{photon} \tag{1}$$

with an error $O(k)$, where k is the energy of the photon, in terms of the physical amplitude for the process

$$A + B \to C + D \tag{2}$$

and the static electromagnetic moments of the particles. It should be remembered that the amplitude for (1) is $O(k^{-1})$ in the limit $k \to 0$, so that two terms of the expansion in powers of k can be calculated from on-shell information. The theorem takes the form

$$M^{\mu} = M_{\mathrm{OS}}{}^{\mu}(T) + O(k), \tag{3}$$

where M^{μ} is the true bremsstrahlung amplitude and $M_{\mathrm{OS}}{}^{\mu}(T)$ is an *on-shell* approximation to it; it is, in fact, a *linear* function of T, the on-shell T matrix for (2).[4] The latter is supposed to be known from elastic scattering experiments.

The original proof[3] was a constructive one, that is, it provided a realization of the theorem in the form of a specific formula for $M_{\mathrm{OS}}{}^{\mu}(T)$. It is clear from the derivation that this formula is not unique, and several others were subsequently written down,[5] all differing by amounts $O(k)$. Any such formula will be designated $M_{\mathrm{OS}}{}^{\mu}(T)$.

Now suppose that the interaction which is responsible for (2) contains a short-range part and a long-range part, *and that the latter is known theoretically.* The statement of the extended theorem is

$$M^{\mu} = M_{\mathrm{OS}}{}^{\mu}(T) + [M_L{}^{\mu} - M_{\mathrm{OS}}{}^{\mu}(T_L)] + O(k). \tag{4}$$

The quantity within brackets in Eq. (4) represents a correction to the original version of the theorem. It consists of the difference between the exact bremsstrahlung amplitude calculated from the long-range part of the interaction, $M_L{}^{\mu}$, and the on-shell approximation to that amplitude, $M_{\mathrm{OS}}{}^{\mu}(T_L)$; T_L is the (theoretical) T matrix due to just the long-range part of the interaction. The significance of the correction, of course, resides in the term $O(k)$, called the error, which we expect to be smaller than the error in Eq. (3). We shall return to this point after providing the theorem.

Since the only requirements for the proof[3] are analyticity of the internal emission amplitude at $\vec{k} = 0$ and gauge invariance, both of which apply just as well to the long-range part of the interaction as to the full interaction, one can also write Eq. (3) for the former, and then subtract to obtain Eq. (4). The motivation for carrying out this subtraction is the well-known result from the two-potential problem that the difference $T - T_L$ is characteristic of an interaction with the short range rather than the full range; and furthermore, the error in the on-shell approximation is expected to decrease with decreasing range of the interaction.[3,6]

In the present paper we test the extended theorem on a model problem consisting of two distinguishable spinless particles that interact via a static potential at all distances. Since the exact amplitude M^μ can be calculated in this model, in addition to the three amplitudes on the right-hand side of Eq. (4), so can the error. Ultimately, of course, one will want to test the theorem on real data.

Our procedure for dividing the interaction into short- and long-range parts is based upon the above discussion of the nucleon-nucleon problem. We divide space into an inner region and an outer region sharply at a separation b, and define the long-range part of the interaction to be the known potential for $r > b$, and zero for $r < b$. Clearly b can be chosen to be any distance greater than the minimum one at which the potential is known, and we shall in fact treat b as a variable.

The discussion in the literature[3,6] about the dependence of the error on the range is only qualitative, and we show below that if the interaction contains both repulsive and attractive parts, then the error in Eq. (4) does not decrease monotonically as b is decreased. It could happen, therefore, that choosing b to be the minimum distance at which one knows the potential might not be optimal. It is straightforward to see that, at least in potential theory, there is a better variable for describing the error in Eq. (4). In this domain the bremsstrahlung amplitude is completely determined by the T matrix, off shell as well as on shell; consequently, if the quantity $|T - T_L|$ is small, so must be $|M^\mu - M_L{}^\mu|$ and also $|M_{OS}{}^\mu(T) - M_{OS}{}^\mu(T_L)|$. In applications of the soft-photon theorem, however, off-shell information is not supposed to be available. We were led, therefore, to try the very simple hypothesis that the on-shell difference $|T - T_L|$ be used to determine the optimum value of b.

To test these ideas we have studied two different potentials. For each potential and at various kinematic points we calculate all four amplitudes and the error in Eq. (4) as a function of b, and look at the correlation with $|T - T_L|$.

For spinless particles and coplanar geometry, all the physical information in the bremsstrahlung amplitude is contained in a single complex number. This follows from the existence of only three independent vectors, which are coplanar, from which the vector amplitude must be constructed, and from the additional fact that only the transverse component contributes to the cross section. Thus, the error term in Eq. (4) can be expressed as one complex number, the magnitude of which is shown on our graphs.

For the on-shell-approximation amplitudes in Eq. (4) we took the nonrelativistic limit of the prescription given in Eq. (2.16) of Ref. 3.[7] In that limit the on-shell T matrix and its energy derivative are evaluated at a center-of-mass energy E which is the average of the initial and final energies of relative motion of the two particles, and at a momentum transfer equal to that which the uncharged particle experiences.

At stated above, for any assumed potential V the long-range potential V_L is equal to V outside the distance b, and exactly zero inside. T_L was computed as a function of b by numerically solving the partial-wave Schrödinger equation for the phase shifts δ_l, and inserting them into

$$T_L = -\frac{8\pi}{m(mE)^{1/2}} \sum_l (2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta).$$

m is the common mass of the two particles and was taken to be the nucleon mass. T is the value of T_L at $b=0$, because the long-range potential then becomes the entire potential.

The exact amplitudes $\vec{M}$ and $\vec{M}_L$ in Eq. (4) were computed as in Heller and Rich,[8] except that the Coulomb potential and spin were not included. We carried out various checks which indicated that each term in Eq. (4) has enough accuracy to obtain a meaningful error term. In one of these checks we compared M^μ with $M_{OS}{}^\mu(T)$ near the limit that the photon energy vanishes, and obtained agreement to about 0.1%.

With the exception of the transformation of the kinematical variables from the laboratory frame to the center-of-mass frame, which was relativistic, the entire calculation was carried out nonrelativistically.

(1) Purely attractive potential.—For our first example, we took the potential to be a single Yukawa function, $V = -V_0\exp(-\mu r)/\mu r$, where $V_0 = 50$ MeV and $\mu = 1.4\ \mathrm{fm}^{-1}$. Results for this potential are shown in Fig. 1, where we have plotted the magnitude of the error and $|T - T_L|$ ver-

sus b. Both functions decrease monotonically as more of the potential is included in V_L. Note that the value of $|\text{error}|$ at $b=\infty$ is precisely the error in the original theorem, Eq. (3). The coplanar geometry used in all our examples is shown as the inset in Fig. 1; particle 1 carries the charge and the two particles emerge at equal laboratory angles to the beam.

(2) Potential with attractive and repulsive parts.—The Reid soft-core singlet S potential[9] was used in all states with $l \leq 3$ at 50 MeV laboratory energy and $l \leq 4$ at 100 MeV. In contrast to the single Yukawa potential, the Reid potential has attractive and repulsive parts, with the repulsion dominating at short range; it is plotted in Fig. 2. Although $|T - T_L|$ and $|\text{error}|$ have much more structure here than for the single Yukawa potential, the striking feature of Fig. 2 is how closely these two quantities track each other. They both start to decrease as b decreases, go through minima at $b \sim 1$ fm, have maxima at $b \sim 0.6$ fm, and then go to zero as b vanishes. We see similar behavior at many other (but not all, see below) kinematic points and it represents a fair confirmation of the hypothesis presented above, namely, that the on-shell difference $|T - T_L|$ is a good variable for describing the error in Eq. (4).

One possible prescription for the use of the extended theorem can now be stated. Calculate the T_L matrix for the long-range part of the potential, plot $|T - T_L|$ as a function of b over the entire range of distances for which the potential is known, and find its minimum value. Finally, at the value of b which gives the minimum calculate

$$M_{OS}^{\mu}(T) + M_L^{\mu} - M_{OS}^{\mu}(T_L)$$

as an approximation to the bremsstrahlung amplitude. For the examples shown in Figs. 1 and 2 (and many others that we have tested) the magnitude of the error in the extended theorem is smaller than that in the original theorem, often by a significant factor. In Fig. 2(a), for example, if one knows the potential down to 1 fm, then the prescription we have given tells you to evaluate the extended theorem at $b = 1.1$ fm where $|T - T_L|$ has its minimum. At that point $|\text{error}|$ is about $\frac{1}{3}$ its value at $b=\infty$. If, on the other hand, the potential were only known down to 1.2 fm, where $|T - T_L|$ is still decreasing, then b should also be chosen as 1.2 fm, at which point $|\text{error}|$ is about $\frac{2}{3}$ its value at $b=\infty$. In Fig. 2(b) the improvement resulting from the extended theorem can be as large as a factor of 6.

The reason for the structure in $|T - T_L|$ in Fig. 2 is well known in nuclear physics. For a particle having small kinetic energy, 50 MeV

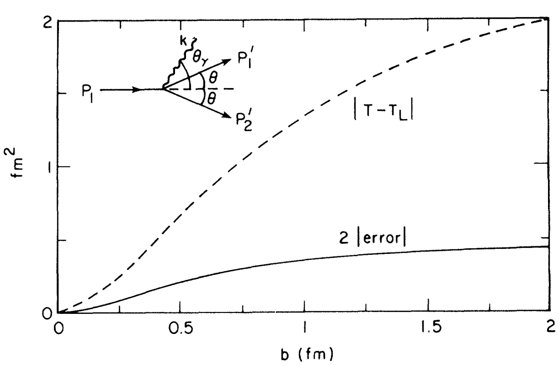

FIG. 1. Results for the single Yukawa potential given in example 1 of the text, at a laboratory kinetic energy $E_L = 50$ MeV, and laboratory angles $\theta = 30°$ and $\theta_\gamma = 5°$. The solid curve shows the magnitude of the error term in Eq. (4) multiplied by 2. The dashed curve shows the magnitude of the difference between the on-shell T matrices T and T_L. Both quantities are displayed as functions of the distance b discussed in the text. The inset defines the angles θ and θ_γ in the laboratory frame.

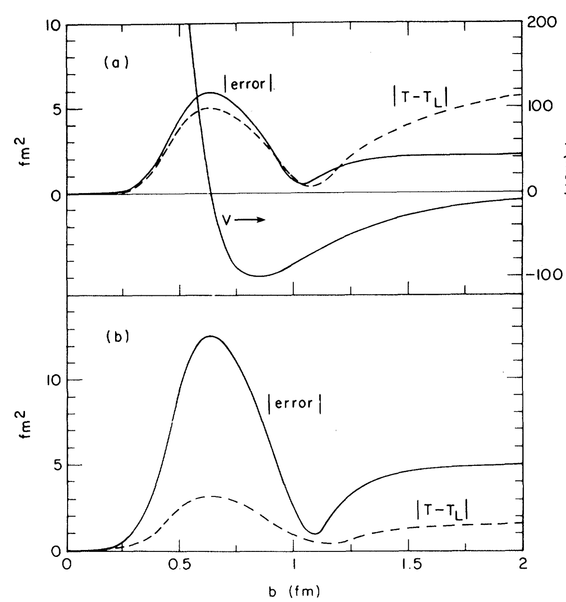

FIG. 2. Results for the Reid soft-core singlet S potential. The potential is shown on the right-hand scale. The solid curves show the magnitude of the error in Eq. (4), and the dashed curves represent $|T-T_L|$. (a) $E_L = 50$ MeV, $\theta = 30°$, and $\theta_\gamma = 5°$. The error at $b=\infty$ is 6.4% of the exact amplitude. (b) $E_L = 100$ MeV, $\theta = 10°$, and $\theta_\gamma = 35°$. The error at $b=\infty$ is 32% of the exact amplitude.

for example, the net effect of the attractive and repulsive parts of the potential is attractive. As b starts to decrease from infinity, the long-range potential starts to become attractive and $|T - T_L|$ decreases. But as b decreases beyond ~1.1 fm, the long-range potential becomes more attractive than the net effect of the complete potential, and $|T - T_L|$ starts to increase. This continues until $b \sim 0.6$ fm at which point the potential becomes repulsive, and then $|T - T_L|$ decreases again. The remarkable result is how closely the error in Eq. (4) follows this same pattern.

In the course of examining the range of validity of the $|T - T_L|$ prescription, we came across a very small kinematic region in which the original version of the theorem is superior to the extended theorem. At a laboratory energy of 100 MeV, with $\theta = 30°$ and $\theta_\gamma = 10°$, the original theorem just happens to be extremely accurate, within 0.8% of the exact result. The effect of decreasing b is to increase the error rather than decrease it, but even so it is only 3.2% at the minimum of $|T - T_L|$. It is not surprising that such a region should exist. Indeed, there is probably a kinematic point at which the original theorem is exact, since that requires the vanishing of a complex number (the error) and there are two real variables θ and θ_γ upon which that number depends (for fixed E_L).

In this paper we have tested the extended theorem on a model problem in potential theory, using a particular on-shell approximation.[3] At many kinematic points it represents a significant improvement in accuracy over the original theorem; and in the small kinematic regions where it does not, it still gives a good approximation to the bremsstrahlung amplitude. We expect similar improvement for other choices of the on-shell approximation, and also when the theorem is applied to experimental data.

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