Resistance of Random Walks

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Unrestricted lattice random walks in which a unit conductor is placed along each bond traversed are considered. The mean end-to-end resistance is studied as a function of the number of steps in the walk and the spatial dimension. A critical scaling law is found whose exponent is consistently given by four different calculational schemes.

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The properties of random walks have been studied for years, both for their intrinsic mathematical interest and for their application to a variety of physical problems.[1] In this Letter, we present the results of a study of the *resistance* of random walks, a property which has received little or no consideration in the literature. We consider an unrestricted random walk of N steps on a hypercubic lattice in d dimensions, and suppose that adjacent lattice sites are connected by a conductance q, where q is the number of times the bond in question is traversed in the walk. We ask how the mean resistance between the end points of the walk, $\langle R(N;d)\rangle$, varies with N. The results of four independent calculational methods, direct Monte Carlo averaging, series summation, an exact solution for the Cayley tree,[2] and a Hamiltonian renormalization-group (RG) analysis, are all consistent with the following picture. In general

$$\langle R(N;d)\rangle \sim N^{x_1(d)}. \tag{1}$$

For $d>4$, $x_1(d)=1$. At $d=4$, there are logarithmic corrections, $\langle R(N,4)\rangle \sim N/(\ln N)^{1/2}$. As d is reduced from 4, $x_1(d)$ decreases initially as $d/4$ and goes to zero at $d=1$, where $\langle R(N;1)\rangle \sim \ln N$. The exponent in (1) appears to be universal, with respect to both lattice topology and certain features of the resistance distribution.

A principal application of our results is for the study of transport in porous sedimentary rocks of low porosity. It has been observed[3] that such materials can retain a connected void space down to extremely low values of porosity, a fact inconsistent with a common approach to microscopically disordered random media[4] in which the pore space is modeled as a percolation network and becomes disconnected at a much larger porosity. The pore space is presumably correlated, and a random walk is one model which guarantees its connectedness. Both electrical conductivity and fluid permeability[5] may be treated in this way.

A heuristic evaluation of the conductivity exponent can be obtained by use of a model similar to the "nodes and links" picture of the spanning percolation cluster.[6] We assume that the pore space consists of a network of winding quasi-one-dimensional strands, treated as random-walk trajectories, meeting at pointlike nodes separated by a characteristic length ξ. If $R(\xi)$ is the resistance of a strand with end-to-end separation ξ, then the conductivity of the material is $\Sigma \sim [R(\xi)\xi^{d-2}]^{-1}$. Using (1) with $\xi \sim N^{1/2}$, we have $\Sigma \sim \xi^{-[d-2+2x_1(d)]}$. The porosity is $\varphi \sim N\xi^{-d} \sim \xi^{2-d}$, whence we obtain Archie's law,[3] $\Sigma \sim \varphi^m$, with $m=[d-2+2x_1(d)]/(d-2)$. In three dimensions this estimate yields $m \cong \frac{5}{2}$; an analogous calculation for a self-avoiding walk would give $m \cong 2$. Although these results are consistent with the data cited in Ref. 3, we regard this calculation as illustrative rather than definitive.

Numerical results.—We have computed the end-to-end resistance of random walks of a fixed number of steps $N=32, 64, 128, \ldots, 4096$, in various spatial dimensions $d=1, 2, \ldots, 5$, by direct numerical simulation. The resulting distributions of resistance are invariably *broad*. For example, in Fig. 1 we show the probability distribution functions for three-dimensional walks of 128 and 4096 steps, after 5000 realizations of the walk. No significant change in shape was noted as the number of realizations varied between 3000 and 10 000, and most results quoted below refer to 5000 walks. There is no discernable tendency for the distribution to sharpen up with increasing N.

In Fig. 2 we show the variation of average resistance with walk length for various d; the sampling error in the mean value is comparable to the size of the dots. For $d=2$ and 3 we fitted these curves by a single power as in Eq. (1) and give the exponent $x_1(d)$ in Table I. The one-dimensional results are less obviously a single

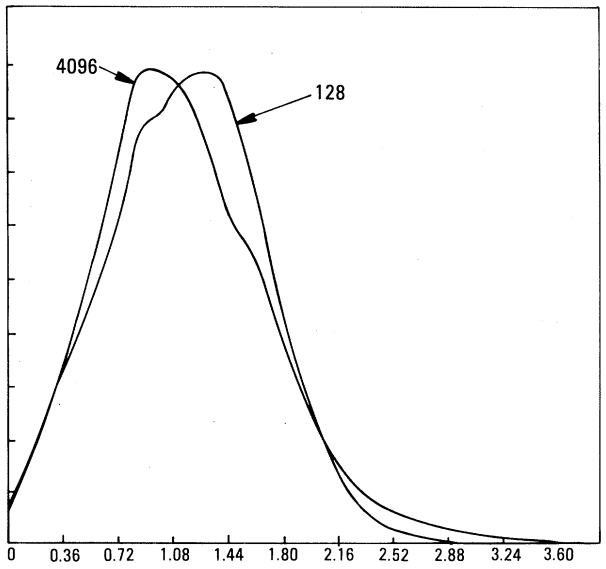

FIG. 1. Probability distribution function of resistance for 128 and 4096 steps in three dimensions; the ordinate is the resistance normalized to $\langle R(N,3)\rangle$.

power than those in higher dimensions, and in fact are consistent with our exact result for the Cayley tree of coordination number 2: $\langle R(N,1)\rangle \sim \ln N$. Likewise the results for $d=4$ are equally consistent with Eq. (1) with $x_1(4)=0.91\pm0.05$ and with the form suggested by the renormalization-group considerations below,

$$\langle R(N;4)\rangle \sim N(\ln N)^{\theta}, \tag{2}$$

with $\theta=-0.55\pm0.05$.

The mean square resistance can also be fitted by a power law,

$$\langle R^2(N;d)\rangle \sim N^{x_2(d)} \tag{3}$$

and the results for x_2 are given in Table I. Error bounds on x_2 are not available, but are presumably comparable to those for x_1. The fact that $x_2(d)\cong 2x_1(d)$ is related to the fact that the width of the probability distribution in Fig. 1 shows little N dependence.

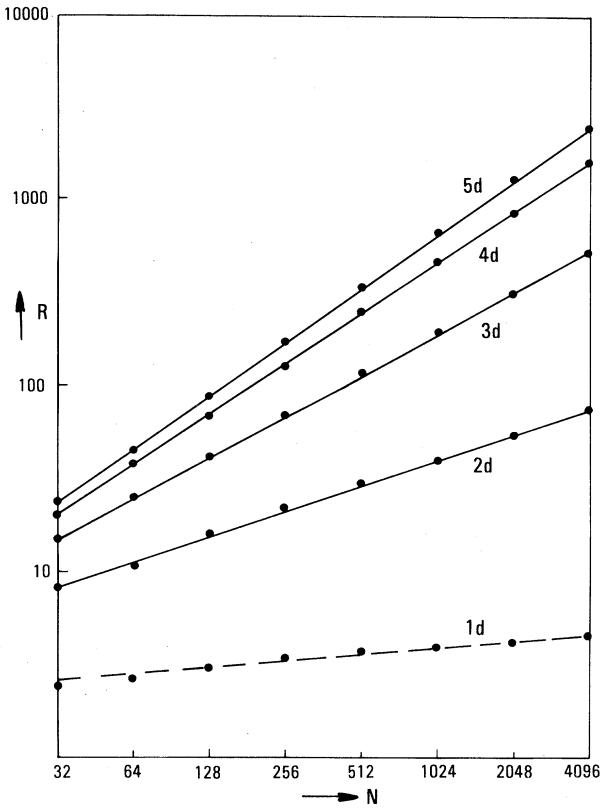

FIG. 2. Mean resistance vs length of walk for various d.

Series analysis.—We define the susceptibilities

$$\chi_l(K)=\sum_{N=1}^{\infty}\sum_{m=1}^{z^N}(R_{N,m})^l K^N, \tag{4}$$

where $R_{N,m}$ represents the resistance of the mth random walk of N steps on a d-dimensional lattice and z is the coordination number. For $l=0$, χ is the random-walk generating function, whereas for $l=1$ and $l=2$, χ is the resistance and resistance squared generating functions, respectively. From the properties of random walks $\chi_l(K)$ is known to have a singularity of the form

$$\chi_l(K)\sim(1-zK)^{-\gamma_l} \tag{5}$$

as $K\to K_c=1/z$ and, of course, γ_0 is equal to 1. The exponents x_l ($l=1,2$) are then related to the

TABLE I. Resistance exponents defined in Eqs. (1) and (3) as a function of d. The RG results are to order ϵ.

	x_1			x_2	
d	Numerical	Series	RG	Numerical	Series
2	0.46 ± 0.01	0.54 ± 0.04	0.5	0.94	1.1 ± 0.1
3	0.73 ± 0.005	0.76 ± 0.02	0.75	1.50	1.51 ± 0.04

corresponding γ's by the relation $x_l = \gamma_l - 1$.

The susceptibility series were computed to order K^{10} as a function of lattice dimensionality for a hypercubic lattice. We checked that the total number of walks of order N is indeed equal to z^N, and for $d=1$ that $\chi_1(K)$ reduces to the exact result.

The series were analyzed with use of the method of inhomogeneous differential approximants.[7] Results for $d=2$ and 3 are given in Table I. For d near $d_c=4$, it is necessary, following Van Dyke and Camp,[8] to take into account corrections to scaling. We used the Rudnick-Nelson[9] form $\chi_1 = AR^{\theta}t^{-\gamma_1}$ with $t=1-zK$ and

$$R = t^{\epsilon/2} + (c/\epsilon)(1 - t^{\epsilon/2}). \quad (6)$$

The renormalization-group analysis (discussed below) yields $\theta = -\frac{1}{2}$. We constructed an expansion in powers of K for $\hat{\chi}_1(K) = \chi_1(K)R^{1/2}$, and the constant c was chosen to be 0.82 by requiring that $\gamma_1 = 2$ ($x_1 = 1$) in four dimensions. An analysis of $\hat{\chi}_1(K)$ then gave γ_1 as a function of d (or ϵ) with no other adjustable parameters. The results are consistent with those of the renormalization-group analysis and yield $\gamma_1 \approx 2 - \epsilon/4$, for $0 < \epsilon < 1$.

Field theory.—We express the resistance R_{ij} between nodes at x_i and x_j in a network as a correlation function for a Potts model[10,11] for which the density matrix, ρ, is

$$\rho = \prod_{i<j} \left(1 + \frac{(s-1)}{1+t\sigma_{ij}}\, \vec{\nu}(x_i)\cdot\vec{\nu}(x_j)\right). \quad (7)$$

Here $\vec{\nu}$ is the Potts vector which can point in any of s directions, and σ_{ij} is the conductance associated with the bond connecting sites i and j. Then we have[11]

$$\lim_{t\to 0}\,[\lim_{s\to 0}\langle \vec{\nu}(x_i)\cdot\vec{\nu}(x_j)\rangle] \sim C_{ij} - tR_{ij}, \quad (8)$$

where the angular brackets indicate an average with respect to ρ and here C_{ij} is unity. To generate random walks, we may identify a walk with the expansion of the matrix inverse $[I-\gamma]^{-1}$, where I is the unit matrix and γ is a matrix having unit elements γ_{ij} if and only if sites i and j are nearest neighbors. These two ideas are married by writing the following Hamiltonian:

$$H = \frac{1}{2}\sum_{\alpha=1}^{n}\sum_{x}[u_\alpha(x)]^2 - \frac{1}{2}K\sum_{\alpha=1}^{n}\sum_{x,x'} u_\alpha(x)\gamma_{x,x'}u_\alpha(x')\prod_{\beta=1}^{m}\left[1+\frac{(s-1)}{(1+t)}\,\vec{\nu}_\beta(x)\cdot\vec{\nu}_\beta(x')\right]. \quad (9)$$

Here we introduce replica indices $\alpha = 1, 2, \ldots, n$ for the random walk variables u_α and independent replica indices $\beta = 1, 2, \ldots, m$ for the Potts model, and the limit $m\to 0$, $n\to 0$ is implied. The resistance correlation functions can be expressed in terms of thermodynamic correlation functions for the Hamiltonian H:

$$\chi_0(x,x') = \langle u_1(x)u_1(x')\rangle$$
$$\chi_1(x,x') = \frac{\partial}{\partial t}\langle u_1(x)u_1(x')\,\vec{\nu}_1(x)\cdot\vec{\nu}_1(x')\rangle. \quad (10)$$

A field theory is generated by introducing Gaussian fields conjugate to $u_\alpha(x)$, $u_\alpha(x)\vec{\nu}_\beta(x)$, $u_\alpha(x)\times\vec{\nu}_\beta(x)\vec{\nu}_\gamma(x)$, and so forth, as in Stephen and Grest.[12] The resulting theory is somewhat similar to that for the "Q" model treated by Priest and Lubensky[13] and there are two distinct fourth-order potentials similar to their $w_1\,\mathrm{Tr}Q^4$ and $w_2[\mathrm{Tr}Q^2]^2$. There are no cubic terms in the field theory because the Gaussian variable u cannot occur an odd number of times.

To first order in $\epsilon = 4-d$ the recursion relations for this field theory are

$$r_k' = b^2[r_k + 8K_d \ln b\,(w_2 r_k + w_1 r_0)],$$
$$w_1' = b^{\epsilon}[w_1 - 16K_d \ln b\,(2w_1^2 + 3w_1 w_2)], \quad (11)$$
$$w_2' = b^{\epsilon}[w_2 - 16K_d \ln b\,(w_1 w_2 + 2w_2^2)],$$

in the notation of Bruce, Droz, and Aharony.[14] Here r_k is the coefficient of the quadratic term in the field theory associated with the variable conjugate to $u_\alpha\prod_{j=1}^{k}\vec{\nu}_{\beta j}$. The bare values of w_1 and w_2 satisfy $w_1 + w_2 = 0$ and the recursion relations preserve this property. For $d > 4$ these recursion relations predict Gaussian behavior, i.e., $x_1 = 1$ in Eq. (1), in agreement with the exact solution for the Cayley tree of coordination number ≥ 3. For $d < 4$ the stable fixed point describing the resistance of random walks has $w_2^* = -w_1^* = \epsilon/16K_d$. The bare value of r_k is $r_k = (1 - zK + kt)/zK$. Thus for $t=0$, all the r_k's are equal and they scale like Gaussian variables as expected for random walks. Turning on t splits the degeneracy of the r_k's so that $(r_l - r_0)' = b^{2-y}(r_l - r_0)$ with $y = \epsilon/2$. Simple scaling arguments show that the exponent x_1 introduced above is given by $x_1 = 1 - y/2$, so that $x_1 = d/4$. Corrections of order ϵ^2 will be given elsewhere.

Universality.—We have verified numerically that the scaling-law exponent is independent of the coordination number of the lattice. If a probability distribution is used to generate the resistances of individual bonds, we find no change in the scaling law. Another model for resis-

tances of random walks can be constructed in which each set of overlapping resistances is replaced by *one* unit resistance. For $d \geq 2$ this model is found, with use of both numerical calculations and the series analysis, to be in the same universality class as the original model. This suggests that loops rather than overlapping resistances are relevant in determining the exponent. In $d = 1$, however, the loop and the overlapping resistances become indistinguishable. Their removal in the second model causes the two models to differ in $d = 1$: $\langle R(N, 1)\rangle \sim \ln N$ for the first model and $\langle R(N, 1)\rangle \sim N^{1/2}$ for the second.

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[1]See, for example, M. N. Barber and B. W. Ninham, *Random and Restricted Walks: Theory and Applications* (Gordon and Breach, New York, 1970); G. H. Weiss and R. J. Rubin, Adv. Chem. Phys. 52, 363 (1982).

[2]A. B. Harris, to be published.

[3]P. N. Sen, C. Scala, and M. H. Cohen, Geophysics 46, 781 (1981).

[4]See, for example, *Ill-Condensed Matter*, edited by R. Balian *et al.*, 1978 Les Houches Summer School Lectures (North-Holland, Amsterdam, 1979).

[5]I. Fatt, Trans. AIME 207, 160 (1956); A. E. Scheidegger, *Physics of Flow through Porous Media* (University of Toronto Press, 1974); J. F. Willemsen, unpublished.

[6]S. Kirkpatrick, in *Electrical Transport and Optical Properties of Inhomogeneous Media—1977*, edited by J. C. Garland and D. B. Tanner, AIP Conference Proceedings No. 40 (American Institute of Physics, New York, 1978).

[7]J. Chen and M. E. Fisher, J. Phys. A 14, 2553 (1981).

[8]J. P. Van Dyke and W. J. Camp, in *Magnetism and Magnetic Materials—1975*, edited by J. J. Becker, G. H. Lander, and J. J. Rhyne, AIP Conference Proceedings No. 29 (American Institute of Physics, New York, 1976), p. 502.

[9]J. Rudnick and D. R. Nelson, Phys. Rev. B 13, 2208 (1976).

[10]C. M. Fortuin and P. W. Kasteleyn, Physica (Utrecht) 57, 536 (1972).

[11]C. Dasgupta, A. B. Harris, and T. C. Lubensky, Phys. Rev. B 17, 1375 (1978); T. C. Lubensky, in Ref. 5.

[12]M. J. Stephen and G. Grest, Phys. Rev. Lett. 38, 567 (1977).

[13]R. G. Priest and T. C. Lubensky, Phys. Rev. B 13, 4159 (1976), and 14, 5125(E) (1976).

[14]A. D. Bruce, M. Droz, and A. Aharony, J. Phys. C 7, 3673 (1974).