Upper Limit on the Frequency of Pulsars

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Neutron stars for which the ratio $T/|W|$ of kinetic to gravitational potential energy exceeds ~ 0.08 are likely to be unstable and to spin down by gravitational-wave–driven oscillations. A class of pulsars with nearly identical frequencies may thus result from the collapse of accreting, rapidly rotating white dwarfs. It is likely that the fast pulsar, PSR 1937 + 214, rotates more slowly than this, but the question remains open partly because of uncertainty in the equation of state.

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It has long been hoped that some neutron stars, when first formed, would rotate sufficiently rapidly to be unstable to nonaxisymmetric perturbations, thus providing an unusually strong source of gravitational waves.[1] There is little evidence that under ordinary conditions such rapid rotation results from stellar collapse; but the apparent discovery of rapidly rotating dwarfs driven by accretion[2] and, recently, of the millisecond pulsar,[3,4] PSR 1937+214, suggests that a neutron star formed when the mass of an accreting dwarf reaches its Chandrasekhar limit can initially be unstable. Because the gravitational radiation emitted by an unstable nonaxisymmetric mode will carry off angular momentum, the final rotation of neutron stars is in principle limited by this instability, which, as discussed below, appears to be important for much smaller values of the ratio $t = T/|W|$ of rotational to gravitational potential energy than is the case for white dwarfs.

Two striking features should characterize the class of dwarf-descended neutron stars, if gravitational radiation is in fact responsible for limiting their rotation—if, that is, the magnetic field is as ineffective in spinning down the star as seems to be the case for PSR 1937+214. First, if the accretion which drives a fast dwarf persists after collapse (or if an old neutron star is spun up by accretion[4]), one would expect to find the fastest neutron stars hovering at the point of nonaxisymmetric instability: The system should radiate gravitational waves whose angular momentum would precisely dissipate that gained in accretion.[5] On the other hand, those neutron stars which free themselves from their source of accretion when they form would have a *single limiting frequency* of rotation: Their masses would be nearly identical, corresponding to the upper mass limit of their dwarf progenitors, and thus their points of instability would be nearly identical as well.

Several accreting dwarfs have observed periods of about 30 s, and, partly because of the remarkable stability of DQ Her ($\dot{P} \sim 10^{-12}$), the periods are generally interpreted as rotation.[2] Dwarfs this fast cannot collapse to neutron stars without ridding themselves of angular momentum, presumably by shedding mass, because their rotation would exceed the breakup speed. A neutron star formed from such a dwarf would initially rotate faster than the instability limit and, if its magnetic field were small, it would again ultimately find itself at the point of marginal stability. (However, its mass would be less than that of its dwarf progenitor, and unlike pulsars formed by "clean" collapse it would not belong to a clearly identifiable class of objects with identical frequencies.)

The discovery that gravitational radiation could induce a nonaxisymmetric instability in rotating stars is due to Chandrasekhar,[6] who found that the Maclaurin sequence (of uniform density, uniformly rotating Newtonian configurations) is unstable beyond the Dedekind bifurcation point, where the frequency of the $l = m = 2$ "bar" mode vanishes. A more detailed analysis of compressible, differentially rotating configurations showed[7] that gravitational radiation makes *all* rotating, self-gravitating perfect fluids unstable to nonaxisymmetric perturbations. Slowly rotating fluids are unstable only to modes with extremely long growth times and short wavelengths (having angular behavior of the form $e^{im\varphi}$ with m large). As one spins up a perfect fluid model, it becomes unstable to modes with successively shorter growth times (and smaller values of m). For white dwarfs, only the $m = 2$ mode grows quickly enough to be significant and only differentially rotating dwarfs can spin fast enough to be unstable to it. For neutron stars, however, modes with $m \lesssim 5$ could grow rapidly enough to be important and, because the equation of state is quite stiff, the instability appears to limit the frequency of uniformly rotating configurations as well.

We will see below that for modes with $m > 5$ (and probably for $m = 5$ as well) the viscous damping time in neutron stars is shorter than the growth time of the instability in a corresponding perfect fluid model. Work by Detweiler and Lindblom[8] on the $l = m = 2$ mode of Maclaurin, by Comins[9] on all the $l = m$ modes of Maclaurin, and by Lindblom and Hiscock[8] on the general class of imperfect relativistic stars shows that when dissipation due to viscosity is comparable to the loss of energy due to gravitational radiation, viscosity will damp out a gravitational-wave-driven instability. Thus for neutron stars, viscosity can be expected to stabilize all modes with $m \geqslant 5$, and instability of an $m = 4$ mode appears to set the limit on rotation.

These points of instability have not yet been calculated for realistic neutron stars models. For white dwarfs, however, work by Ostriker and co-workers[10] showed that the value $t = 0.14$ of $T/|W|$ at which the $l = m = 2$ mode of Maclaurin becomes unstable was surprisingly insensitive to the compressibility and to the rotation law of the star. Although the tensor virial method they used turned out to provide neither a necessary[11] nor sufficient[7,12] condition for instability, when Durisen and Imamura[13] recomputed the instability point using a genuinely sufficient condition, they again found only small ($< 7\%$) departures from the Maclaurin value of t. Neutron stars are substantially stiffer than dwarfs [for the range of models given by Arnett and Bowers[14] the index $\gamma = 1 + n^{-1} = p(\rho + p)^{-1}\, dp/d\rho$ is in the range $2 \leqslant \gamma \leqslant 3$], and in this sense they resemble the Maclaurin models more closely than do dwarfs. One therefore again expects the Maclaurin sequence to provide an approximate guide to the value of t at instability points of the long-wavelength modes. In Table I the values of t at which modes with $l, m \leqslant 5$ become unstable to radiation along the Maclaurin sequence are given. The inaccuracy in extrapolating these results to neutron stars should reflect the difference between the (real part of the) normal-mode frequencies in Newtonian and in relativistic stars having the same central density and adiabatic index. A comparison by Balbinski and Schutz[15] of the $l = 2$ mode in corresponding Newtonian and relativistic models gives only a 9% difference in real frequencies when $GM/Rc^2 \sim 0.3$ and the values of t in Table I may provide a guide of similar accuracy to neutron-star instability points.

TABLE I. Instability points along the Maclaurin sequence: zero-frequency modes in an inertial frame. Here $t = T/|W|$ and e denotes the eccentricity; for $(l, m) = (4, 2)$, $(3, 2)$, $(3, 1)$, and $(2, 1)$, no zero-frequency mode occurs.

l, m	t	e
6, 6	0.0531	0.576
5, 5	0.0629	0.617
4, 4	0.0771	0.668
4, 3	0.162	0.851
4, 1	0.178	0.873
3, 3	0.0991	0.731
2, 2	0.138	0.813

A formalism adequate to compute the instability points of Maclaurin has been available since 1889.[16] Instability to gravitational radiation, however, sets in via a mode with zero frequency in the inertial frame, and these points held no interest to early investigators because instability to viscosity sets in via a mode with frequency $\omega = m\Omega$ (zero frequency in the rotating frame). Thus zero-frequency modes were apparently not investigated until Comins[9] looked at the $l = m$ modes to verify directly that all of the Maclaurin sequence was formally unstable to gravitational radiation and to elucidate the combined effect of viscosity and gravitational radiation on these modes. He did not find exact instability points, but rough values of the eccentricity (for the $l = m$ modes) can be extrapolated from his tables and agree with the values given in Table I. Table I was constructed by iteratively locating the zeros of the frequency equation given by Roberts and Stewartson.[16,17]

From the table it is clear that the $l \neq m$ modes become unstable only for large values of t—in fact, only after the bar mode is itself unstable—and at that point the bar mode dominates. Consequently one need only consider $l = m$ modes, whose e-folding times are given in the post-Newtonian approximation by Comins's second paper.[9] From Eq. (3e) there, we have[18]

$$\tau = \frac{(m-1)[(2m+1)!!]^2}{(m+1)(m+2)(1-e^2)^{1/2}} \left(\frac{c}{R\omega}\right)^{2m+1} \frac{\omega + (m-1)\Omega}{2\pi G\rho},$$

where ρ is the density, Ω the angular velocity, R the radius, and e the eccentricity of the model. Writing $K = (\pi G\rho)^{-1/2}[\omega + (m-1)\Omega]$ and using for

e its value at the instability point gives

$$\tau(m=2)=16\,\frac{K}{(\pi G\rho)^{1/2}}\left(\frac{c}{R\omega}\right)^5,$$

$$\tau(m=3)=\frac{800K}{(\pi G\rho)^{1/2}}\left(\frac{c}{R\omega}\right)^7,$$

$$\tau(m=4)=\frac{6\times10^4K}{(\pi G\rho)^{1/2}}\left(\frac{c}{R\omega}\right)^9,$$

$$\tau(m=5)=\frac{7\times10^6K}{(\pi G\rho)^{1/2}}\left(\frac{c}{R\omega}\right)^{11}.$$

Typical $1.4M_\odot$ models in Arnett and Bowers[14] have $(\pi G\rho)^{1/2}\sim10^4$ s^{-1} and $R\sim10$ km, and when t is within 0.03 of its value at the instability point of a Maclaurin mode, $\omega\lesssim\frac{1}{2}(\pi G\rho)^{1/2}$. Then, setting $K=1$ and $\omega=0.5\times10^4$ s^{-1}, we have the following rough values for the growth times; values corresponding to $\omega=10^4$ s^{-1} are given in parentheses to illustrate the large uncertainty arising from the exponent of ω: $\tau(m=2)\sim10$ s (0.4 s), $\tau(m=3)\sim2\times10^4$ s (2×10^2 s), $\tau(m=4)\sim6\times10^7$ s (10^5 s), $\tau(m=5)\sim2\times10^{11}$ s (10^8 s). Thus, with each successive mode, the growth time increases by a factor of at least 10^3 (see also Ref. 5).

The damping time associated with shear viscosity[9] is roughly $\tau_\nu=(m-1)^{-1}(2m+1)^{-1}R^2\nu^{-1}$. The kinematic viscosity ν depends strongly on temperature ($\nu\sim T^{-2}$), except possibly in the crust, where impurities may dominate electron scattering.[19] In the first years after formation,[20] $T\gtrsim10^9$ K in the interior and 1 cm^2 g$^{-1}<\nu\lesssim100$ cm^2 g^{-1} throughout the star. Then 10^8 s $<\tau_\nu<10^{11}$ s, probably stabilizing the $l=m=5$ mode (certainly stabilizing all higher modes) and leaving the $l=m=4$ mode as the most likely candidate to set the limit on rotation.[21]

From Table I then, the point of instability is at $t\sim0.08$. To interpret this for relativistic stars I will adopt the customary definition of the gravitational potential energy, namely $|W|=(M_p-M)c^2$, where

$$M_p\equiv\int\rho(1-2GM/rc^2)^{-1/2}4\pi r^2dr$$

is the proper mass and M is the gravitational mass. Then $t=\frac{1}{2}I\Omega^2/(M_p-M)c^2$, and the frequency $\Omega/2\pi$ corresponding to $t=0.08$ is given in hertz by

$$\Omega/2\pi=2.7\times10^3\left|\frac{(M_p-M)/M_\odot}{I/10^{45}\text{ g cm}^2}\right|^{1/2}.$$

On the expectation that dwarf-descended neutron stars will have the baryon number of a $1.4M_\odot$ dwarf, Table II gives values of $\Omega/2\pi$ for each of the Arnett-Bowers models whose baryon mass, M_A, is $1.4M_\odot$.

TABLE II. Frequencies of Arnett-Bowers models for $T/|W|=0.08$. (Asterisks indicate extrapolation to $M_A=1.4M_\odot$.)

Arnett-Bowers table (model)	$M_A/M_\odot$	$M/M_\odot$	$\Omega/2\pi$ (Hz)
10 (M)	1.283	1.211	677
	1.4		707*
12 (O)	1.408	1.279	914
11 (N)	1.404	1.280	950
4 (C)	1.4		1060*
	1.436	1.324	1077
5 (D)	1.4		1215*
	1.444	1.313	1234
6 (E)	1.404	1.266	1316
7 (F)	1.400	1.269	1341
2 (A)	1.404	1.272	1385
3 (B)	1.401	1.248	1868
8 (G)	1.392	1.254	1980

The limiting frequency of rotation is clearly strongly dependent on the equation of state used, but the dependence on mass is relatively small, with $d\ln\Omega/d\ln m\sim1$. The 642-Hz frequency of the fast pulsar lies just below the range of Ω for proposed equations of state. However, the Arnett-Bowers models are spherical, and the increase in I due to rotation may lower Ω by about 15%. [For Newtonian stars of comparable stiffness, $I(t=0.08)\cong1.3I(t=0)$ and $W(t=0.08)\cong0.95W(t=0)$,[22] from which the estimated decrease in Ω follows.] Then 642 Hz would be higher than $\Omega/2\pi$ for model M and within 20% of $\Omega/2\pi$ for models N and O.

One further caveat should be mentioned: The role of the crust has so far been ignored. If it oscillates as an elastic solid, it would be unlikely to alter substantially the picture sketched here, but as the amplitude of a mode grows, the crust is likely to break, perhaps into plates whose boundaries are the nodes. Subsequent interactions (plate collisions, for example) might dissipate enough energy to damp out the instability, effectively limiting the pulsation amplitude. A newly formed neutron star should spin down in seconds to $t\sim0.14$ via the $l=m=2$ mode and, as the crust hardens, in hours or days via the $l=m=3$ mode. If, however, the amplitude of the $l=m=4$ mode were sharply limited by the crust, the spin-down time to $t\sim0.08$ could vastly exceed the mode's growth time (of a few years). Then, as the star cooled, the viscosity should quickly increase, damping the instability within 100–1000 yr [when $T \lesssim 10^8$ K in the interior, ν can be as large as 10^4–10^5 cm^2 g^{-1} (Ref. 19)].

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