Gravitational Radiation Reaction in the Binary Pulsar and the Quadrupole-Formula Controversy

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The evolution of the orbit of a binary pulsar under the action of gravitational radiation reaction is calculated. No approximation is made of weak gravity inside the individual stars; the details of the orbital motion are given directly (to order c^{-5} and G^3). The calculation reveals no acceleration of the center of mass of the system, and a secular decrease of the time of return to periastron. The quantitative results agree both with the well-known "quadrupole formula" and with observations.

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In recent years, the discovery of binary pulsars[1] and the report[2] of the measurement of a secular acceleration of the orbital motion of PSR 1913 + 16 has fueled a lively controversy[3] about the applicability to such systems of several "quadrupole formulas" which, on one hand, have been derived[4] from general relativity with very unequal levels of rigor and completeness, and which, on the other hand, have both different physical meanings ("gravitational energy flux" at infinity, "radiation reaction," ...) and different domains of validity. Postponing the discussion of the pure mathematical rigor of the existing derivations (none being absolutely satisfactory from this point of view), I wish to emphasize here that none of the existing derivations, except the one outlined below, meet, at the same time, the two following requirements which are, however, indispensable if one wishes to compare the theoretical predictions with the observations[1,2]: (i) The derivation should apply to *compact* bodies (radius $\sim Gm/c^2$, which implies strong internal gravity), and (ii) one should compute the *direct* effect of the nonlinear retarded gravitational interaction on the absolute orbital motion of a member of a binary system. Few attempts have been aimed at meeting the latter requirement.[5]

The derivation presented here relies on recent results[6-8] based on a new method[9] especially tailored for computing the third-post-Minkowskian gravitational field outside two compact bodies and for deducing therefrom (by an improved Einstein-Infeld-Hoffmann-Kerr-type approach) the equations of motion of the two bodies. The latter, manifestly Poincaré-invariant, retarded functional equations of motion have been computed and then transformed into ordinary differential equations while keeping *all* the post-Newtonian corrections up to the fifth order in c^{-1} where time-irreversible effects show up. This was achieved by carrying out the first (order G), second[6,7] (order G^2), and third[8,9] (order G^3) iterations of Einstein's equations. The necessity of considering the third iteration for dealing with gravitationally *bound* systems had been known for a long time.[10] It has been shown recently[7,8] that the same is true even when computing the net mechanical energy loss during *small-angle scattering*. For both cases the correct mechanical energy loss was first derived in Ref. 8 and shown to confirm the expected "quadrupole formula." However, as stressed in Ref. 9, even such a *mechanical* energy-loss formula is still grossly insufficient for controlling the full kinematical behavior of the binary system as is done below.

A full knowledge of the orbital motion of a binary system can be obtained in three steps: (1) reducing this two-body problem to a one-body problem; (2) solving the latter one-body problem when the terms of order c^{-5} are neglected (these terms will be seen, *a posteriori*, to play the role of "radiation reaction terms"); and (3) solving the full problem by means of the method of variation of arbitrary constants. Some technical details of this approach, which makes use of previous results (Damour and co-workers[6-9]), follow; the final result is discussed in the last two paragraphs.

The first step towards solving the equations of motion, complete up to order c^{-5}, of a binary system[7,8] consists in proving a generalization of the *center-of-mass theorem* valid up to order c^{-5} *inclusive*. Such a generalization has been shown to hold, up to order c^{-4}, in Ref. 7, where six first integrals (up to c^{-4}) of the binary system, $P_{(4)}{}^i$ and $K_{(4)}{}^i$, generalizing the total linear momentum and the center-of-mass constant, were constructed ($i = 1, 2, 3$). If we introduce

$$P_5{}^i = \tfrac{4}{5} G^2 mm'(m - m')[V^2 - 2G(m + m')/R]R^{-3}Z^i$$

and

$$G_5{}^i = -\tfrac{4}{5} Gmm'(m-m')[V^2 - 2G(m+m')/R]\,V^i/(m+m')$$

($Z^i = z^i - z'^i$, $V^i = v^i - v'^i$, $R = |Z^i|$), a straightforward calculation shows that $P_{(5)}{}^i \equiv P_{(4)}{}^i + c^{-5}P_5{}^i$ and $K_{(5)}{}^i \equiv K_{(4)}{}^i + c^{-5}(G_5{}^i - tP_5{}^i)$ are constant (modulo c^{-6}). Then, by a suitable Poincaré transformation, we can choose a frame of reference where both $P_{(5)}$ and $K_{(5)}$ are zero (center-of-mass frame). In this frame one can express the position and velocity of the second body (the "companion") in terms of the position and velocity z and v of the first body (the "pulsar"). In this frame one can write autonomous equations for the motion of the first body:

$$dv/dt = B_0(z) + c^{-2}B_2(z,v) + c^{-4}B_4(z,v) + c^{-5}B_5(z,v) + O(c^{-6}), \tag{1}$$

where $B_0{}^i(z) = -Gm'^3(m+m')^{-2}|z|^{-3}z^i$. In order to integrate Eq. (1) we start by considering the following auxiliary differential system:

$$dv/dt = B_0(z) + c^{-2}B_2(z,v) + c^{-4}B_4(z,v). \tag{2}$$

It has been shown[7] that the equations of motion of the *binary* system admitted, under *neglect* of terms of order c^{-5}, ten first integrals: $P_{(4)}{}^i$, $K_{(4)}{}^i$ (quoted above), and $E_{(4)}(z, z', v, v')$, $J_{(4)}{}^i(z, z', v, v')$. Replacing in the last four quantities z' and v' by their "center-of-mass" expressions in terms of z and v and subtracting all the terms of formal order c^{-5}, we end up with four first integrals (modulo c^{-6}) of the differential system (2): $c_1 = \hat{E}_{(4)}(z,v)$, $c_2{}^i = \hat{J}_{(4)}{}^i(z,v)$. Moreover the explicit vectorial structure of $c_2{}^i$ shows that the (fictitious) motion (2) takes place in a plane. We can therefore introduce polar coordinates r, w such that $z^i = (r\cos w,\ r\sin w,\ 0)$. The knowledge of the first integrals c_1 and $c_2 = (c_2{}^i c_2{}^i)^{1/2}$ yields the following equations for $r(t)$ and $w(t)$:

$$(dr/dt)^2 = R(r, c_1, c_2), \tag{3}$$

$$dw/dt = G(r, c_1, c_2), \tag{4}$$

where R and G are polynomials of the fifth degree in $1/r$ whose coefficients are polynomial in c_1 and c_2 and rational in m and m'. Equations (3) and (4) can be solved by means of two quadratures. In order to take full advantage of this solution [to Eq. (2)] for integrating Eq. (1), it is convenient to introduce several new quantities and functions. One can prove that if $2(m+m')c_1c_2{}^2 < -G^2(mm')^3$ the polynomial $r^5R(r)$ admits two and only two real roots, $r_1 < r_2$, which have nonzero limits when $c^{-1} \to 0$. Let us then define

$$P(c_1, c_2) = 2\int_{r_1}^{r_2} dr[R(r, c_1, c_2)]^{-1/2},$$

$$P(c_1, c_2)/Q(c_1, c_2) = \pi^{-1}\int_{r_1}^{r_2} dr\,G(r, c_1, c_2)[R(r, c_1, c_2)]^{-1/2}.$$

Let us also define a *hyperelliptic function* $S(l, c_a)$ ($a = 1, 2$) by inverting the incomplete hyperelliptic integral: $l = (2\pi/P)\int_{r_1}^{S} dr\,R^{-1/2}$. Moreover let

$$W(l, c_a) = (P/2\pi)\int_0^l dx\{G(S(x, c_a), c_a) - 2\pi/Q\}.$$

Both functions $S(l)$ and $W(l)$ are C^∞ and periodic in l (period 2π). The solution to Eqs. (3) and (4) can be written as

$$r = S(l, c_1, c_2), \tag{5}$$

$$w = m + W(l, c_1, c_2), \tag{6}$$

with $l = 2\pi t/P(c_1, c_2) + c_0$, and $m = 2\pi t/Q(c_1, c_2) + c_3$. In this manner we have succeeded in expressing the solution of the auxiliary system (2) in terms of the time t and of four *constants* c_A ($A = 0, 1, 2, 3$). We can now integrate the actual equations of motion (1) by applying an "improved" method of variation of arbitrary constants. We look for solutions to Eq. (1) which have the functional form (5) and (6) (together with the corresponding velocity expressions) but with variable $c_A(t)$ and with the following "improved" expressions for the "relativistic anomalies" l and m:

$$l = 2\pi\int_0^t du[P(c_1(u), c_2(u))]^{-1} + c_0(t), \tag{7}$$

$$m = 2\pi\int_0^t du[Q(c_1(u), c_2(u))]^{-1} + c_3(t). \tag{8}$$

One can then prove that this is possible and that the $c_A(t)$ have to satisfy some evolution equations of the type $dc_A/dt = c^{-5}F_A(l, c_B)$ where the functions $F_A(l)$ are *periodic* in l (period 2π). Therefore if we restrict our attention to the evolution of the c_A's over a time scale $\ll c^5P/v^5$, we can prove that each $c_A(t)$ is equal to a term periodic in time [period $P(c_a{}^0)$] and a secular term, *linear* in time: k_At, where each k_A is given by a complete hyperelliptic integral:

$$[2/P(c_a{}^0)]\int_{r_1}^{r_2} dr\,K_A(r, c_a{}^0)[R(r, c_a{}^0)]^{-1/2};$$

$$c_a{}^0 \equiv c_a(0).$$

The knowledge of the k_A's, together with Eqs. (5)–(8), is sufficient for determining the *secular* effects in the orbital motion. For instance from Eq. (5) and the definition of the function $S(l)$ one sees that r will reach its (slowly changing) minimum value r_1, i.e., that the first object of the binary system will pass through its periastron, each time $l(t)$ is equal to a multiple of 2π. It is then easy to deduce from Eq. (7) that the date of the Nth periastron passage is

$$t_N = t_0 + (P_0^{-1} + k_0/2\pi)^{-1}N + \tfrac{1}{2}P_0\dot{P}_0N^2,$$

where $P_0 \equiv P(c_1^0, c_2^0)$ and where

$$\dot{P}_0 \equiv k_1 \frac{\partial P(c_1^0, c_2^0)}{\partial c_1^0} + k_2 \frac{\partial P(c_1^0, c_2^0)}{\partial c_2^0}.$$

The explicit computation of $\dot{P}_0$ leads to evaluation of several complete hyperelliptic integrals. However, one can prove rigorously that the latter integrals are well approximated (when $v \ll c$) by simpler circular integrals. The final result is

$$\dot{P}_0 = -\frac{192\pi}{5c^5}\left(\frac{2\pi G}{P_0}\right)^{5/3}\frac{mm'}{(m+m')^{1/3}}\left(1+\tfrac{73}{24}e_0^2+\tfrac{37}{96}e_0^4\right)(1-e_0^2)^{-7/2}, \tag{9}$$

where e_0 denotes $[1+2(m+m')c_1^0(c_2^0)^2/G^2(mm')^3]^{1/2}$. It is to be stressed that in this approach[9] each parameter m or m' is the "Schwarzschild mass" of each compact object, when isolated, and not the integral of some Newtonian density which would be a poor numerical approximation to the "Schwarzschild mass." This is one of the reasons why most of the post-Newtonian derivations of the "quadrupole formula" are of doubtful applicability to the case at hand.

Because it has been proved above that the center of mass of the system is unaccelerated and because in this post-Minkowskian approach[9] the coordinate time t is a proper time far away from the system, we can conclude that the theoretical quantity $\dot{P}_0$, Eq. (9), which measures the secular decrease of the time of return to the periastron, must be identified (modulo a constant Doppler factor close to unity) with the observational quantity denoted by $\dot{P}_b$ in Ref. 2. Similarly it can be proved that the present e_0 is, within the accuracy now available, to be identified with the observational parameter e. The same is true for the two masses: $m = m_p$, $m' = m_c$.

I have therefore proved by directly solving the equations of motion of two compact bodies in general relativity that the *absolute* orbital motion of each of the bodies should exhibit, when seen from far away, the secular acceleration $\dot{P}_0$ given by Eq. (9). This result agrees both with the standard, heuristically predicted[11,12] "quadrupole formula" and with the observations of the Hulse-Taylor pulsar.[2,4] Note, however, that this derivation, the details of which will be published elsewhere, never had to make use of such concepts as quadrupole moment, energy *flux* at infinity, balance equations, energy, angular momentum, or radiation damping force.

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