Chaos in the Semiclassical N-Atom Jaynes-Cummings Model: Failure of the Rotating-Wave Approximation

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The problem of a collection of two-level atoms interacting with a single-mode classical electromagnetic field is considered. It is found that the model with an initial population inversion exhibits chaotic behavior that becomes stronger for higher atomic number density. The chaos is characterized by Fourier analysis and the maximal Lyapunov exponent. In the rotating-wave approximation, however, there is no prediction of chaos.

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One of the fundamental theoretical paradigms of quantum optics is the Jaynes-Cummings model.[1] As originally formulated, the model consists of a two-state atom interacting with a single mode of the electromagnetic field. The virtue of the model is that, within the rotating-wave approximation (RWA), it is exactly solvable, regardless of whether the field is treated classically or quantum mechanically. A generalization of this model to include N two-state atoms interacting with a single mode of the electromagnetic field was studied by Tavis and Cummings.[2] Again, with the RWA it is exactly solvable.

The RWA is used almost universally in quantum optics, and its validity is seldom questioned. An exact treatment, if possible, typically leads in resonance problems to only very small quantitative differences from the RWA. An example is the Bloch-Siegert shift.[3] Tavis and Cummings[2] note that the breakdown in the RWA occurs for extremely high-intensity fields. In fact, it is just in this regime where the atomic system can generate high-intensity fields that we observe chaos and the failure of the RWA.

In this Letter we reconsider the model of Tavis and Cummings, a system of N atoms per unit volume interacting with a single-mode field. We treat their fully quantum problem semiclassically (i.e., no field quantization), but do not restrict ourselves to the RWA. We show that this fundamental model of quantum optics admits chaotic behavior. However, when the RWA is made, only quasiperiodic solutions are found.

The interaction of a two-state atom with an electric field $E(t)$ is described by the well-known optical Bloch equations[3]:

$$\dot{x}(t) = -\omega_0 y, \tag{1a}$$

$$\dot{y}(t) = \omega_0 x + (2p/\hbar)E(t)z(t), \tag{1b}$$

$$\dot{z}(t) = -(2p/\hbar)E(t)y(t). \tag{1c}$$

With a density N of atoms, the single-mode field of frequency ω satisfies the Maxwell equation

$$\ddot{E}(t) + \omega^2 E(t) = -4\pi N p \ddot{x}(t). \tag{2}$$

We are using a fairly standard notation in which p is the transition dipole moment, assumed to be real. ω_0 is the transition frequency of each atom, z is the population difference, and x and y are the dipole variables defined in terms of off-diagonal density-matrix elements. Let $\tau = \omega_0 t$, $\tilde{E}(t) = (2p/\hbar\omega_0)E(t)$. Then with τ as the independent variable, we may write (1) and (2) as

$$\dot{x}(\tau) = -y(\tau), \tag{3a}$$

$$\dot{y}(\tau) = x(\tau) + \tilde{E}(\tau)z(\tau), \tag{3b}$$

$$\dot{z}(\tau) = -\tilde{E}(\tau)y(\tau), \tag{3c}$$

$$\ddot{\tilde{E}}(\tau) + \mu^2\tilde{E}(\tau) = \beta\dot{y}(\tau), \tag{3d}$$

where we have introduced the dimensionless parameters

$$\mu = \omega/\omega_0, \quad \beta = 8\pi N p^2/\hbar\omega_0. \tag{4}$$

Taking $p = 1$ D and $\omega_0 = 10^{15}$ Hz, we have $\beta = 2.4 \times 10^{-23} N$, where N is given in units of inverse cubic centimeters.

Equations (3) have been solved numerically for various values of μ and β and for different initial conditions. For our purposes here it is sufficient to focus on the particular case of exact resonance ($\mu = 1$) and the initial conditions $x(0) = y(0) = 0$, $z(0) = \pm 1$ for the atomic variables. In other words, all the atoms are excited ($z = +1$) or in their ground state ($z = -1$) at $t = 0$. For the case of no population inversion we do not find chaotic behavior. This occurs physically because our initial conditions for the electric field are such that the RWA initially is a valid approximation, $\tilde{E}(0) = 10^{-6}$, and $\dot{\tilde{E}}(0) = 0$. If the initial conditions for the atoms allows a population inversion ($z > 0$) and there are a sufficiently large number of atoms, then this initially small electric field can grow to high intensities, the RWA will fail, and chaos will be exhibited. We have found that the better the RWA is in describing the dynamics, the weaker the degree of chaotic behavior.

Let us now consider the physically interesting regime where initially $z=+1$, $\tilde{E}(0)=10^{-6}$, and $\dot{\tilde{E}}(0)=0$ as a function of the atomic density through the parameter β. Figures 1 and 2 show $z(\tau)$, together with the power spectrum of this time series obtained by applying a cosine bell window and then taking a 4096-point fast Fourier transform. As β increases we see increasingly greater complexity in the spectrum. Figure 2, for instance, strongly suggests that the system is chaotic for $\beta=1$.

The dynamical system (3) with $\beta=0$ is manifestly orderly (nonchaotic), for the integral $x^2+y^2+z^2=1$ reduces it to a system of order two. Figure 3(a) is a plot of x vs y for this case. In Fig. 3(b) we show the corresponding result for $\beta=1.0$. The orderly pattern in the case $\beta=0$ has been destroyed.

Chaotic behavior is characterized by "very sensitive dependence on initial conditions." In particular, it is generally recognized that a rigorous measure of chaotic behavior may be given in terms of the Lyapunov spectrum of the system. A positive characteristic Lyapunov exponent is associated with chaotic behavior because on average it represents an exponential separation of initially close trajectories. For the purpose of simply identifying chaotic behavior it is sufficient to compute the largest exponent, and this may be done by use of the technique described by Benettin, Galgani, and Strelcyn.[4] The result of the computation for the parameters of Fig. 2 gives $\lambda \simeq 0.087$, showing that the atom-field system in this case exhibits chaos. For small values of β we appear to be converging to smaller values $\lambda>0$, whereas for $\beta=0$, $\lambda=0$. Thus the chaotic behavior of the system appears to be-

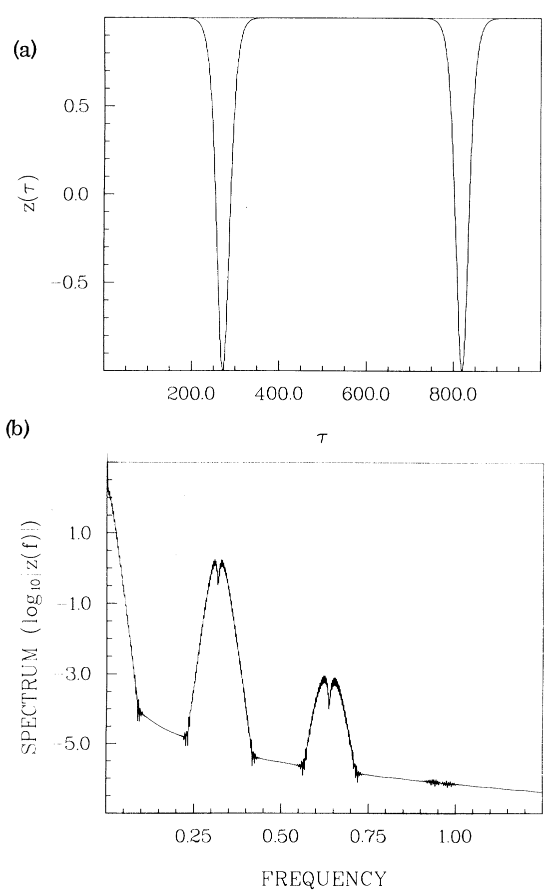

FIG. 1. (a) Population inversion $z(\tau)$ for $\beta=0.01$. (b) Power spectrum of the time series shown in (a).

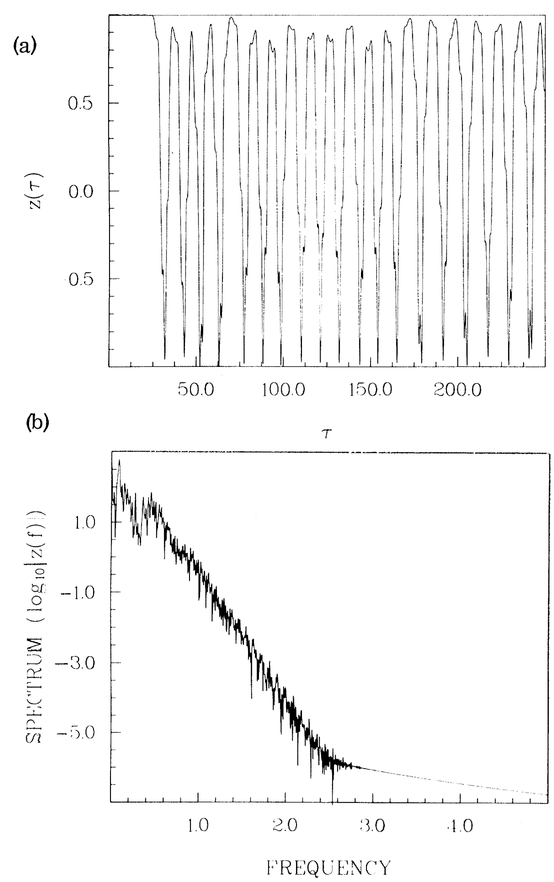

FIG. 2. (a) Population inversion $z(\tau)$ for $\beta=1.0$. (b) Power spectrum of the time series shown in (a).

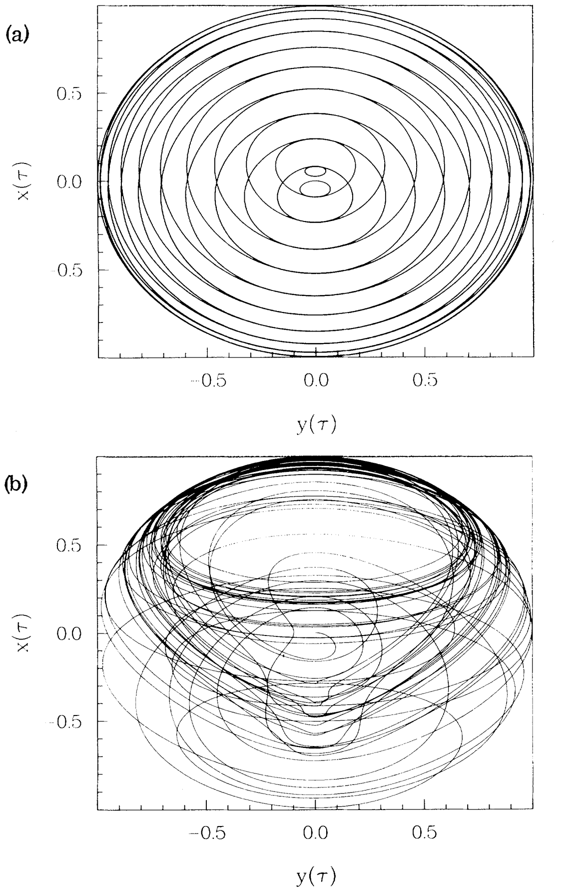

FIG. 3. $x(\tau)$ vs $y(\tau)$ for (a) $\beta = 0$ and (b) $\beta = 1.0$ where $\tilde{E}(0) = 1$.

come more pronounced as β increases.

In the RWA we write

$$x(t) = u(t)\cos\omega t - v(t)\sin\omega t, \tag{5a}$$

$$y(t) = u(t)\sin\omega t + v(t)\cos\omega t\,, \tag{5b}$$

$$z(t) = w(t)\,, \tag{5c}$$

$$E(t) = \epsilon(t)\cos\omega t, \tag{5d}$$

and assume that u, v, w, and ϵ are slowly varying compared with $\cos\omega t$. Then the RWA version of (1) and (2) is found to be[3]

$$\dot{u}(t) = -\Delta v(t)\,, \tag{6a}$$

$$\dot{v}(t) = \Delta u(t) + (p/\hbar)\epsilon(t)w(t)\,, \tag{6b}$$

$$\dot{w}(t) = -(p/\hbar)\epsilon(t)v(t)\,, \tag{6c}$$

$$\dot{\epsilon}(t) = 2pN\pi\omega v(t)\,. \tag{6d}$$

This system has two integrals: $u^2 + v^2 + w^2 = \text{const}$ and $N\hbar\omega w + \epsilon^2/4\pi = \text{const}$, and these reduce the dimension of the autonomous system (6) to two. In such a case the RWA equations can give only orderly time dynamics [analytical solutions of (6) have been known for some time].

However, the integrals of (6) break down when damping terms are included. In fact the system (6) with $\Delta = 0$ and damping terms included is isomorphic to the Lorenz model, as previously noted by Haken.[5] Thus, with damping terms included, the RWA equations of the semiclassical Tavis-Cummings model admit chaos for a certain range of parameter values.[6]

The fact that the RWA version (6) of the system (1) precludes chaotic evolution suggests that caution should be exercised in other quantum-optical problems where the possibility of chaotic behavior exists. This is not the only example in which a well-known approximation fails in this context: If one includes damping terms in (6) and then makes the rate-equation approximation, the order of the system is reduced from three to two, and the new dynamical system has only regular behavior.

A more detailed analysis of this model of chaotic behavior in quantum optics will be given elsewhere. At this point it may be useful to summarize our main results: (A) The system (3) exhibits chaotic behavior, but its RWA version (6) does not. (B) The degree of chaos observed in (3) is inversely related to the validity of the RWA as we have found by observing the time development of the magnitude of the variable $\tilde{E}(\tau)$. (C) The chaos becomes more pronounced as β increases, but it may nevertheless be present for all $\beta > 0$. (D) Chaotic behavior is obtained for certain parameter ranges if damping terms are included in the RWA equations (6).

The initial field value $\tilde{E}(0) = 10^{-6}$ assumed in Figs. 1 and 2 corresponds to an initial Rabi frequency of 10^{-6} times the transition circular frequency, a physically reasonable value. Actually our conclusions are insensitive to $\tilde{E}(0)$. The values $\beta = 0.01$ and 1.0 correspond to $N = (4.17$ and $417) \times 10^{20}$ atoms/cm^3 for $p = 1$ D. Fairly large number densities are therefore required to realize the strongly chaotic regime. When N is large, the field generated by the atoms becomes large, and, as we previously stated, this is just the regime where the RWA fails. For lower values of N, the RWA may give an excellent approximation for the oscillation amplitudes, while nevertheless giving orderly rather than chaotic evolution. In such a case there is little relative energy in the broadband, chaotic portion of the time evolution.

The problem of extending a classical or semiclassical analysis of chaotic behavior to quantum mechanics is of great interest. Within the RWA neither the semiclassical nor fully quantized treatments predict chaos,[7] whereas we have shown that without the RWA the semiclassical theory does predict chaos. It would therefore be very interesting to consider the fully quantized Tavis-Cummings model without the RWA. This problem is presently under investigation.

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[1]E. T. Jaynes and F. W. Cummings, Proc. IEEE 51, 89 (1963).

[2]M. Tavis and F. W. Cummings, Phys. Rev. 170, 379 (1968).

[3]See, for example, L. Allen and J. H. Eberly, *Optical Resonance and Two-Level Atoms* (Wiley, New York, 1975).

[4]G. Benettin, L. Galgani, and J. M. Strelcyn, Phys. Rev. A 14, 2338 (1976); G. Benettin and L. Galgani, J. Stat. Phys. 27, 153 (1982).

[5]H. Haken, Phys. Lett. 53A, 77 (1975).

[6]In the case of a single-mode, homogeneously broadened laser transition, the Lorenz-type chaos occurs only if the field loss rate exceeds the sum of the transverse and longitudinal decay rates, an atypical situation.

[7]A detailed analysis of the fully quantized RWA Jaynes-Cummings model has been reported by J. H. Eberly *et al.*, Phys. Rev. Lett. 44, 1323 (1980), and Phys. Rev. A 23, 236 (1981), and J. Phys. A 14, 1383 (1981). Though complicated, the dynamics in this case is not chaotic in the sense meant here (i.e., a positive characteristic Lyapunov exponent implying sensitive dependence on initial conditions). (J. H. Eberly, private communication.)