Excitations of SU(5) Monopoles

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The author reinterprets Witten's analysis of the dyons in the SU(2) Georgi-Glashow model, showing that his peculiar formula for their electric charge is due to a symmetry of the semiclassical configuration space under the noncompact group of real numbers. This method of analysis extends easily to the SU(5) dyons and shows that their color hypercharge is also arbitrary when the vacuum angle θ is nonzero.

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In this Letter I shall discuss the semiclassical quantization of the fundamental SU(5) monopoles of Dokos and Tomaras. To begin, let us briefly recall the application of the method to two systems in quantum mechanics, both with classical energy $E=\frac{1}{2}\dot{r}^2+(1/e^2)V(er)$, where V is as shown in Figs. 1(a) and 1(b). The first system is of interest because it models the behavior of the vacuum states in gauge theories, while the second will turn out to be more appropriate to the low-lying monopole states. Since the two systems are qualitatively different, the nature of the corresponding quantum states will be different; in particular the electric charge and color hypercharge of the grand unified monopoles will be unquantized.

Canonically quantizing example one yields a Hilbert space of states spanned by the position eigenstates $|x\rangle$. If we denote by C, the "semiclassical configuration space," the set of all minima of V, then $C=\{x_i\}$ is in 1-to-1 correspondence with Z, the set of integers. The semiclassical method now tells us that the low-lying energy eigenstates are well approximated as $e\to 0$ by linear combinations of the $|x_i\rangle$, $x_i\in C$, and that furthermore the best choices of these combinations are those that diagonalize any exact symmetries of the Hamiltonian.

We have characterized C as a set of points, but what is its topology? Which of its points should be regarded as "very close" to one another? The answer is that *none* of the points are close; i.e., the discrete topology is appropriate for C. This is because for fixed e and T, $\langle x_i|e^{-HT}|x_j\rangle$ does not approach 1 as x_j ranges through $C-\{x_i\}$: Between any two points there is a finite Euclidean action barrier. Furthermore, since the symmetry group of H is also Z acting by discrete translations, the low-lying energy states are the Bloch waves, representations of Z labeled by an angular "crystal momentum": $|\theta\rangle=\sum_n e^{in\theta}|x_n\rangle$.

Now consider Fig. 1(b). This example differs from the first in that (a) one direction in the potential is flat, and (b) some absolute minima of V are separated from others by barriers requiring infinite action to traverse. While C thus consists of the points on the two troughs of V, it is consistent to restrict to just one when finding the stationary states of the full quantum theory. Call C' the points along one trough and $I_{C'}$ those along the other. These are related by the inversion operator I along the x axis.

Since distinct points of C' can now be close in the sense described above, the topology of C' is not discrete. Again not surprisingly we find that C' is homeomorphic to R, the real numbers. Since the symmetry group of translations of C' is also R, the low-energy eigenstates are characterized by an arbitrary real number p: $|p\rangle$

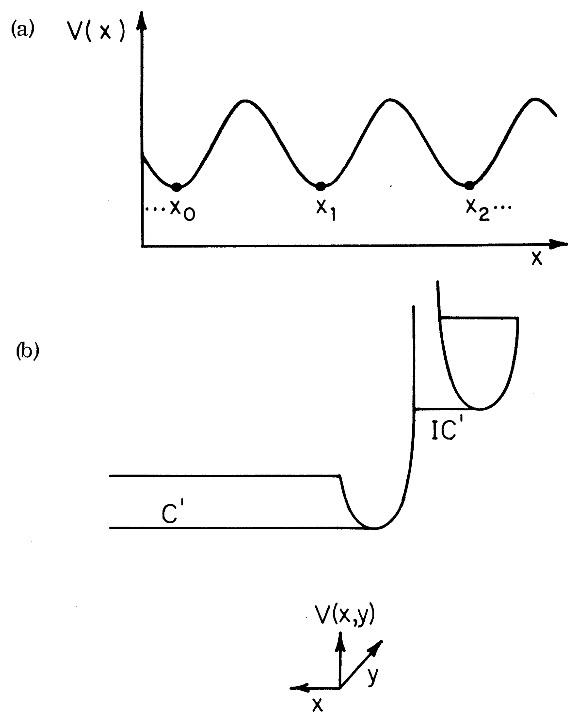

FIG. 1. Potentials for two classical systems.

$=\int dx\, e^{ipx}|x\rangle$, $x\in C'$. The splittings between the states are proportional to the eigenvalues of the Laplacian on R: $E\propto p^2$. The eigenvalue p is not quantized, because of the noncompactness of the associated symmetry group.

We have now found two sets of approximate energy eigenstates of the complete system, namely $|p\rangle$ and $I|p\rangle$. While we are not compelled to assemble these into parity eigenstates $\frac{1}{2}(1+I)|p\rangle$, neither is this forbidden; the infiniteness of the barrier between the troughs will then be reflected by the lack of any splitting in the energies of the resulting states $|p,\pm\rangle$.

The transcription of these considerations to gauge field theories at weak coupling is straightforward. Consider for example SU(2) broken spontaneously to U(1) by an adjoint Higgs field φ, and quantize in the temporal gauge $A_0=0$. As in the examples, the absolute minima of the classical energy functional are stationary. They may all be described as gauge transformations of $\varphi(\vec{x})\equiv c\tau_3$, $\vec{A}(\vec{x})\equiv 0$ by a function $g(\vec{x})$. Apparently $C=\{\text{maps: } R^3\to SU(2)\}$. In the temporal gauge, however, we must consider physical states averaged over "little" gauge transformations[1] (LGT), i.e., those of the form $\exp[\lambda^a(\vec{x})T^a]$, where T^a are the anti-Hermitian generators and $\lambda^a(\vec{x})\to 0$ as $|\vec{x}|\to\infty$. This means that C should actually be taken as a quotient space under equivalence up to LGT. In addition[1] we can reduce C to C' consisting only of those classes of g which can be reached from $g\equiv 1$ by a path of finite action. Such g must at spatial infinity lie in the stabilizing group stabφ of the Higgs field, since otherwise any interpolating history between $g\equiv 1$ and $g(\vec{x})$ would have $\dot{\varphi}\neq 0$ throughout infinite volume and so would have infinite action. Thus we must have $g(\vec{x})\to\exp[-i\tau_3\lambda(\vec{x})]$ as $|\vec{x}|\to\infty$.

Simple power counting now shows that $\lambda(\vec{x})$ must at infinity approach a constant independent of angles.[2] The value of this constant is in fact irrelevant, as both $g(\vec{x})$ and $g_0 g(\vec{x})$ describe the same vacuum configuration, where g_0 is a constant group element. That is, part of the full symmetry group acts trivially on the vacuum, so that C is finally the maps into SU(2) which approach 1 at infinity modulo LGT. These maps form the group $\pi_3[SU(2)]\cong Z$. The Belavin-Polyakov-Schwartz-Tyupkin inequality[3] now says that the discrete topology for C is appropriate, so that this is example one and the vacuum states are labeled by an angular parameter θ.

The analysis proceeds similarly when we instead minimize E subject to the constraint that a monopole be present. Thanks to the monopole's topological stability,[4] the resulting quantum theory is still consistent. We replace the energy by the "excess energy," the amount by which it exceeds the absolute minimum in this sector, and similarly the action. Let $|M\rangle$ be the eigenstate of the field operators with eigenvalues the radial-gauge 't Hooft–Polyakov monopole,[4] averaged over all LGT. The configurations of minimum energy are again labeled by functions $g: R^3\to SU(2)$.

We can again reduce C to C', in which $g\to\exp(-i\times\vec{\tau}\cdot\hat{r}\pi v)$, $v\in R$. (If the unbroken group were non-Abelian we would also have to demand $g\to\text{stab}\varphi\cap\text{stab}Q$, where Q is the generator of the long-range gauge field, to avert a linear divergence in excess action.) Now, however, we cannot set the constant v equal to 0, since the monopole is not invariant to global charge rotations. g is thus partly specified by its asymptotic value $e^{i\pi v}$, and since any g_1, g_2 with the same $e^{i\pi v}$ differ by $g_1 g_2^{-1}\to 1$, a single integer completes the description of g up to LGT. Thus C' corresponds to $Z\times[0,2\pi)$.

To find the topology of C', choose a representative of each of its classes, say $g^{(v)}=\exp[-i\vec{\tau}\cdot\hat{r}\lambda(r)]$, where $\lambda(0)=0$ and $\lambda(\infty)=\pi v$. It is easy to find an interpolating field path from $g^{(v)}$ at time zero to $g^{(v+\epsilon)}$ at fixed time T with Euclidean action of order ϵ. This means that C' is homeomorphic to R and this is example two. The low-lying monopole excitations are characterized by a real number p, which is not periodic. In particular, the energy $E\propto p^2$.

To see what is going on, note[5] that the unitary operator $U(v)$ implementing $g^{(v)}$ equals $\exp(2\pi i v Q/e)$ when acting on physical states, where Q is the electric charge operator. Since $g^{(1)}$ has winding number[6] equal to 1, we have that $\theta=p \bmod 2\pi$, and the electric charge $q=ep/2\pi=e(n+\theta/2\pi)$, where n is an integer. This is Witten's result. Since $(n,\theta+2\pi)$ and $(n+1,\theta)$ describe the same p, it is no surprise that the respective energies are equal, nor that they are proportional to the square of p, the charge. Nowhere has it been necessary to consider the internal structure of the monopole.

The generalization to SU(5) is immediate. Starting with the fundamental monopole, the minima of E are specified by $g(\vec{x})$ such that there exists

$$g(\hat{x})=\lim_{|\vec{x}|\to\infty} g(\vec{x})$$

$$\subseteq \text{stab}\varphi\cap\text{stab}Q\subseteq G=SU(3)\otimes U(1)_{EM}.$$

The subgroup of G is $SU(2)\otimes U(1)_Y\otimes U(1)_{EM}$, but the SU(2) acts trivially on $|M\rangle$ much as in the vacuum sector. Since the remaining group is Abelian we again have $g(\hat{x})$ independent of angles. Furthermore, one linear combination of the generators $T_Y=-\frac{1}{2}i\lambda_8$ and T_{EM} acts trivially on $|M\rangle$ as well, so that finally we have $C\simeq R$ just as before. The states $|p\rangle$ have electric charge $q=ep/2\pi$, color hypercharge $y=-q\sqrt{3}/4e$, and $B-L=q/2e$, just as the classical Dokos-Tomaras dyons.

In the quantum theory, however, q is arbitrary. At first this seems to be no problem, since when we reduced C to C' we obtained an Abelian symmetry group whose covering group is noncompact. However, the remaining color symmetry generators are analogous to the inversion I of example two, and so we should be able, as stated by Dokos and Tomaras,[7] to form dyons in various SU(3) multiplets. (The I-type rotations should, however, remain in stabφ, since otherwise the I eigenstates would violate cluster decomposition.) In particular, since $|p\rangle$ has definite isospin and hypercharge, the fact that the latter is arbitrary is somewhat nettlesome.

We must conclude that either (a) θ is required to be zero, or else (b) the existing generators of global $SU(2)\otimes U(1)\otimes U(1)$ gauge transformations cannot be extended to a set for the full unbroken $SU(3)\otimes U(1)$. Since such an extension always exists in the absence of a monopole, both options have the unattractive feature that the monopole, a localized object, affects the global nature of the vacuum. Defining the generators as in Ref. 8, I have shown that in fact option (b) is correct, that in the presence of a monopole a homotopy obstruction renders the extension impossible. I will not give the proof here, since subsequently Manchar and I[8] have arrived at a more general result.[9]

Thus there exists a wide variety of dyon states with ordinary charge, color hypercharge, and $B-L$ all proportional to an arbitrary real number p. All are obtained from our $|p\rangle$ states by large gauge transformations. These dyons have no particular color, and hence no Clebsch-Gordan decompositions for the scattering amplitudes of colored particles, even when $\theta=0$.

In an attempt to understand the physical implications of these results, one might argue that for any $\theta\neq 0$ our monopoles, being color charged, will necessarily be confined. If there are no massless quarks this would seem to require that fundamental monopoles be bound to antimonopoles; the subsequent annihilation could account for their observed paucity today. This scenario is probably not correct. According to 't Hooft's conjectured confinement mechanism,[10] in pure QCD the ground state is well described by a set of "transient" dynamical variables: one U(1) gauge field for each of the Cartan generators of SU(3), electrically charged quarks and gluons, and excitations which are magnetic monopoles with respect to the Maxwell fields. This picture should remain valid in the low-energy reduction of a grand unified theory. Even if a genuine SU(5) monopole is introduced we expect no interference with confinement, a local phenomenon. Choosing the Cartan generators to include the monopole's Q, it then has Abelian quantum numbers proportional to those of transient excitations, regardless of θ. The latter condense in the confining phase, and so the SU(5) monopole should be unconfined for small but nonzero θ, unlike quarks and other states with only electric charges. Its color will be screened at the confinement scale by the plasma of transient excitations.

We can repeat the discussion near each of several widely separated monopoles even if their Q do not match. If this argument is valid, then for small $\theta\neq 0$ grand unified monopoles should indeed be unconfined hadron-like particles, just as in the $\theta=0$ case. For finite θ there can be phase transitions in the theory. The continuity prior to these, however, means that the discovery of monopoles could not be used to determine that θ is precisely zero.

Prior to this work Abouelsaood arrived at a conclusion slightly different from mine, that some chromodyon excitations do not exist because of their vanishing moments of inertia.[11] I thank him for showing me his work. I also thank S. Coleman, H. Georgi, A. Manohar, and E. Witten for valuable discussions. This work was supported in part by the National Science Foundation under Grants No. PHY77-22864 and No. PHY82-15249, and by a National Science Foundation graduate fellowship.

[1]C. Callan, R. Dashen, and D. Gross, Phys. Rev. D 17, 2717 (1978).

[2]The situation is more complicated when the unbroken group is non-Abelian. As we will see, the problem does not arise in the monopole sector.

[3]A. Belavin, A. Polyakov, A. Schwartz, and Yu. Tyup-

kin, Phys. Lett. 59B, 85 (1975).
[4]S. Coleman, Harvard University Report No. HUTP-82/A032 (to be published).
[5]E. Witten, Phys. Lett. 86B, 283 (1979); F. Wilczek, Phys. Rev. Lett. 48, 1146 (1982).
[6]S. Coleman, in *The Whys of Subnuclear Physics*, edited by A. Zichichi (Plenum, New York, 1979).
[7]C. Dokos and T. Tomaras, Phys. Rev. D 21, 2940 (1980).
[8]A. Manohar and P. Nelson, following Letter [Phys. Rev. Lett. 50, 943 (1983)].
[9]The original proof is still needed in the relatively uninteresting case of the $3n$-monopole sector, $n \neq 0$. Reference 8 shows that global color generators can be defined, but there is still no paradox since no such set arises as an extension of the original subgroup generators.
[10]G. 't Hooft, Nucl. Phys. B190, 455 (1981).
[11]A. Abouelsaood, Harvard University Report No. HUTP-82/A058 (to be published).