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Fractal Basin Boundaries, Long-Lived Chaotic Transients, and Unstable-Unstable Pair Bifurcation

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A new type of bifurcation to chaos is pointed out and discussed. In this bifurcation two unstable fixed points or periodic orbits are created simultaneously with a strange attractor which has a fractal basin boundary. Chaotic transients associated with the coalescence of the unstable-unstable pair are shown to be extraordinarily long-lived.

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In this paper we consider a new type of bifurcation to chaotic motion. This bifurcation is characterized by the simultaneous appearance of a pair of unstable fixed points or periodic orbits, a fractal (i.e., nondifferentiable) basin boundary, and a chaotic attractor (also commonly called a strange attractor). Just prior to this type of bifurcation, transient behavior with a chaotic character can take place, and *these chaotic transients can be extremely long.* The existence of such remarkably long-lived chaotic transients may have important implications for experiments on chaotic systems.

To motivate our considerations, we note that the general question of how chaotic attractors arise as a system parameter is varied is of great fundamental interest. According to conventional wisdom, only a small number of distinct types of chaotic attractor onsets are generally seen. Among these are period doubling,[1] intermittency,[2] and crises.[3] In the first two a nonchaotic attracting orbit evolves into a chaotic one. On the other hand, in a crisis a chaotic transient converts into a chaotic attractor. As a concrete example of the latter, say that when p, a parameter of the system, is in the range $p > p_*$, a nonchaotic attractor exists. In addition, for $p > p_*$, chaotic transients are also observed to occur before orbits settle into the nonchaotic attractor. As p approaches p_* from above the average duration of a chaotic transient approaches infinity, and past p_* a chaotic attractor appears by conversion of the chaotic transient. For $p < p_*$ the chaotic and nonchaotic attractors coexist, each with its own separate basin of attraction. (The basin of attraction of an attractor is the set of initial conditions whose trajectories asymptotically approach that attractor as time increases.) In general there will be some boundary separating these two basins. The question of how the chaotic attractor is created (as p decreases through p_*), or, inversely, destroyed (as p increases through p_*) has only recently been addressed.[3] Generally, the disappearance of the chaotic attractor occurs via a motion of the attractor and its basin boundary toward each other, with the critical state ($p=p_*$) occurring when the two touch. In previous works this has been studied for the case in which the basin boundary is smooth (cf. Ref. 3 and references therein).

In this paper we demonstrate a new type of bifurcation which causes the birth or death of a chaotic attractor (as at $p=p_*$ in the previous discussion). For the phenomenon in question, the basin boundary is a fractal (nondifferentiable) curve. We find that fractal basin boundaries and our new type of bifurcation to chaos can occur in systems of at least four (autonomous) differential equations, invertible maps of at least three dimensions, or noninvertible maps of at least two dimensions. (An analogous situation holds for chaotic attractors which occur in noninvertible one-dimensional maps but require two dimensions for the invertible case.[4]) To illustrate the phenomena in the simplest context, we shall restrict the present treatment to the two-dimensional noninvertible case.[5] Our work on the three-dimensional invertible case will be reported elsewhere.[6] We believe that, as a general rule, noninvertible m-dimensional maps can be used to model phenomena which can only occur in invertible maps of dimension greater than m.

To begin we wish to consider the occurrence of fractal basin boundaries. To our knowledge, explicit discussion of fractal basin boundaries has been restricted to the study of analytic maps of a single complex variable. While of basic interest, such maps are not typical models of dynamical systems. In particular, by virtue of the Cauchy-Riemann relations, chaotic attractors cannot occur in such maps. To demonstrate a fractal basin boundary in the simplest possible context, consider the following two-dimensional map,

$$\theta_{n+1} = 2\theta_n \bmod 2\pi, \tag{1a}$$

$$z_{n+1} = \lambda z_n + \cos\theta_n, \tag{1b}$$

where we take $2 > \lambda > 1$, and the range of θ is taken to be $0 \leq \theta < 2\pi$. This map has been previously studied by Kaplan and Yorke[7] for $|\lambda| < 1$, in which case it has a chaotic attractor but no fractal basin boundary. Given θ_{n+1}, it is not possible to find θ_n uniquely since there are *two* possible solutions of (1a), $\theta_n = \theta_{n+1}/2$ and $\theta_n = \pi + \theta_{n+1}/2$. Thus the map (1) is noninvertible. The Jacobian matrix of the map, Eqs. (1), has eigenvalues 2 and λ. Since both exceed 1 there can be no attractors with finite z. In fact, almost all initial conditions will generate orbits that are asymptotic to either $z=+\infty$ or $z=-\infty$ as $n\to+\infty$. We consider $z=+\infty$ and $z=-\infty$ (with $0\leq\theta<2\pi$) as two attractors and ask what is their basin boundary. That is, what is $f(\theta)$ such that initial conditions in $z>f(\theta)$ lead to orbits which are asymptotic to $z=+\infty$, while those in $z<f(\theta)$ yield $z\to-\infty$. The calculation of $f(\theta)$ is facilitated by two observations: (i) the dynamics of the θ variable is independent of z [cf. Eq. (1a)], and (ii) since forward iterates of an initial point are repelled by the boundary, backward iterates approach it. Say we pick a value θ and wish to find the corresponding $z=f(\theta)$. We take $\theta=\theta_0$ in Eq. (1a) and iterate θ_0 forward, $\theta_n = 2^n\theta_0 \bmod 2\pi$. Imagine that we iterate forward up to $n=N$. Now choose some value $z' = z_N$ and iterate z backward along the θ orbit orginally generated from θ_0. Use of $\theta_n = 2^n\theta_0 \bmod 2\pi$ together with Eq. (1b) yields an expression for z_0. Letting $N\to\infty$ in this expression, z_0 approaches the basin boundary, and we obtain an equation for the basin boundary,

$$z = f(\theta) = -\sum_{l=1}^{\infty} \lambda^{-(l+1)} \cos(2^l\theta). \tag{2}$$

Since $\lambda > 1$, the sum in (2) converges absolutely and uniformly. Now consider $df/d\theta$. From (2) $df/d\theta = \frac{1}{2}\sum_l (2/\lambda)^{(l+1)} \sin(2^l\theta)$. Since $\lambda < 2$, this sum diverges, and thus $f(\theta)$ is nondifferentiable. In fact, the curve (2) has an infinite length and a fractal dimension d which is between 1 and 2 [it can be shown[8] that $d = 2 - (\ln\lambda)(\ln 2)^{-1}$].

To illustrate our new type of bifurcation to chaos, we consider a modification of Eqs. (1),

$$\theta_{n+1} = 2\theta_n \bmod 2\pi, \tag{3a}$$

$$z_{n+1} = \alpha z_n + z_n^2 + \beta\cos\theta_n. \tag{3b}$$

Figure 1 shows results of iterating Eqs. (3), for a particular set of parameters ($\alpha=0.5$, $\beta=0.04$). For these parameter values there are apparently two attractors, $z=+\infty$ and a chaotic attractor located in the region, $-0.1 \lesssim z \lesssim 0.1$. The solid black region in Fig. 1 is the basin of attraction for the attractor $z=+\infty$. The blank region in Fig. 1 is the basin of attraction for the strange attractor. Also shown in Fig. 1 are a large number of iterates of the map generated by a single initial condition in the blank region. This may be regarded essentially as a picture of the chaotic attractor. (For our purposes, it is useful to regard $z=+\infty$ as representing a general nonchaotic attractor.)

To see why a fractal basin boundary is expected,

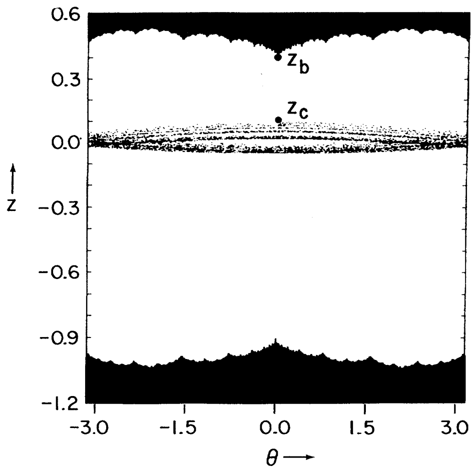

FIG. 1. The attractor and its basin for $\alpha = 0.5$ and $\beta = 0.04$. Points initialized in the dark region lead to orbits which are asymptotic to $z = +\infty$, while points outside the dark region yield orbits asymptotic to the strange attractor. In order to display more clearly the collision of the chaotic attractor and its basin boundary (which occurs at $\theta = 0$), we have plotted the horizontal axis on $(-\pi, \pi)$ rather than $(0, 2\pi)$. In effect we are identifying θ and $\theta - 2\pi$ for $\pi \leq \theta \leq 2\pi$. Thus $\theta = 0$ is at the center of the figure.

consider Fig. 1. We note that the upper basin boundary is located in the region $0.55 \gtrsim z \gtrsim 0.45$. As a crude approximation to the dynamics in this region we set $z = \beta\delta + 0.50$, and linearized (3b) about $z = 0.5$ to obtain $\delta_{n+1} = 1.5\delta_n + \cos\theta_n$, which is the same as Eq. (1b) with $\lambda = 1.5$. The same crude argument can be used to make plausible the presence of a chaotic attractor. Namely, linearization of (3b) about $z = 0$ again yields Eq. (1b) but with $\lambda = 0.5$. Kaplan and Yorke[7] have shown that Eqs. (1) with $|\lambda| < 1$ have a chaotic attractor.

Magnification of a small portion of the basin boundary clearly reveals its fractal structure.[6] The lower basin boundary in the vicinity of $z = -1.0$ is essentially a preimage of the upper basin boundary. Every point in the black region in $z < 0$ (i.e., below the lower basin boundary) maps on one iterate to a point above the upper basin boundary, after which z remains positive and accelerates to $z = +\infty$ [e.g., this can be readily seen to be the case for large negative z, since in this case Eq. (3b) yields $z_{n+1} \cong z_n^2$].

Now consider what happens as α is increased from $\alpha = 0.5$ (the value corresponding to Fig. 1). Observe that the smallest z value on the upper basin boundary (denoted z_b) occurs at $\theta = 0$, and that the largest z value on the chaotic attractor (denoted z_c) also occurs at $\theta = 0$. As α increases, these two points mover closer together, until, at some critical value $\alpha = \alpha_*$, they first touch. We find numerically that $\alpha_* \cong 0.6$. Further, one can show that $(\theta, z) = (0, z_c)$ and $(0, z_b)$ are fixed points of the map. Thus we predict α_* by looking at the fixed points of Eqs. (3). The value $\theta = 0$ automatically satisfies Eq. (3a) for all n. Putting $\theta = 0$ in (3b) and assuming that z_n is independent of n (i.e., a fixed point) yield $z = \alpha z + z^2 + \beta$, or $z_\pm = \{(1-\alpha) \pm [(1-\alpha)^2 - 4\beta]^{1/2}\}/2$, and $z_b = z_+$ and $z_c = z_-$. Linearization of the map (3) about these two fixed points shows that they are both unstable with z_+ strictly repelling and z_- repelling in one direction and attracting in the other (a saddle). As α is increased from $\alpha < 1 - 2\beta^{1/2}$, the two points move toward each and coalesce at $\alpha = 1 - 2\beta^{1/2}$ (cf. equation for $z_\pm$). Past $\alpha = 1 - 2\beta^{1/2}$ the fixed points annihilate and no longer exist. Thus we predict $\alpha_* = 1 - 2\beta^{1/2}$, or $\alpha_* = 0.6$ for $\beta = 0.04$. For $\alpha > \alpha_*$ the chaotic attractor no longer exists and almost *all* initial conditions are eventually attracted to $z = +\infty$. Thus as α is increased through $\alpha = \alpha_*$ the unstable fixed-point pair annihilates and the chaotic attractor along with its basin dies. Conversely, as α decreases through $\alpha = \alpha_*$, the unstable fixed-point pair and the chaotic attractor and its basin are born.[9] [Thus when the attractor dies (or is born) it does so by colliding with an unstable fixed point on the basin boundary. In the terminology of Ref. 3 such an event is called a *crisis.*] Although our example, Eqs. (3), exhibits the phenomenon of unstable-unstable pair bifurcation to chaos for the case in which the pair is a pair of *fixed* points, we emphasize that the same considerations apply when the pair is a pair of *periodic* orbits. We have verified this numerically by examination of other maps [e.g., add a constant phase shift into the cosine term in (3b)].

Now consider what happens when α just exceeds α_*. It is observed that orbits initialized in the region which was formally the basin of attraction for the chaotic attractor are initially drawn to what looks like the old chaotic attractor. The orbit then bounces around on this chaotic attractor remnant in a chaotic fashion, as for $\alpha < \alpha_*$. After some time, however, the orbit lands sufficiently near the region of coalescence of z_+ and z_- and then rapidly leaves the chaotic attractor remnant, accelerating to large positive z values. Thus the chaotic attractor $(\alpha < \alpha_*)$ is replaced by a *chaotic transient* $(\alpha > \alpha_*)$. Figure 2 shows a chaotic transient for $\beta = 0.04$ and $\alpha = 0.65 > \alpha_*$

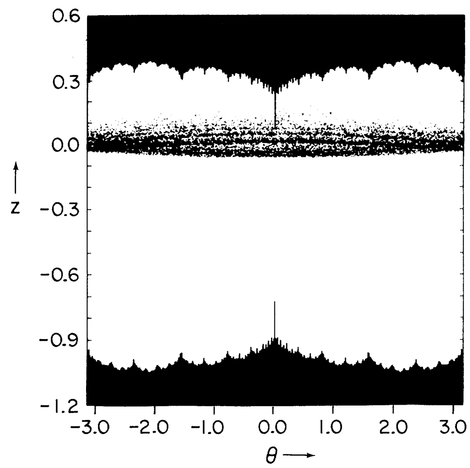

FIG. 2. Picture of a chaotic transient for β = 0.04 and α = 0.65 > α_* = 0.06. See caption to Fig. 1 for the definition of θ.

$=0.60$. The black region represents initial points which rapidly acquire large z values. This region is fairly well defined because the average length of a chaotic transient is very long ($\sim 2 \times 10^7$) for these parameters. Also shown in Fig. 2 are the first 10^4 iterates of a chaotic transient generated from a single initial condition in the blank region. Note the apparent penetration of the black region into the region of the attractor remnant at $\theta = 0$.

We have obtained numerical results for the average lifetime of a chaotic transient (denoted $\langle\tau\rangle$) by averaging over many different randomly chosen initial conditions in the basin of attraction. These results are found to agree well with our theoretical prediction (cf. Ref. 6), $\ln\langle\tau\rangle \sim [(\pi/\beta^{1/4})\ln 2](\alpha - \alpha_*)^{-1/2}$, derived for $\alpha - \alpha_* \ll \alpha_*$. In particular, we have numerically examined the case $\beta = 0.04$ for $\alpha - \alpha_*$ in the range 0.07 to 0.15 and find very good agreement with $\ln\langle\tau\rangle = 4.87(\alpha - \alpha_*)^{-1/2} - 4.53$, where the constant term is a fitting parameter. The remarkable aspect of this result for $\langle\tau\rangle$ is that *these chaotic transients can be very long-lived.* For example, even at a value of $\alpha \sim 20\%$ above α_* the transient is of the order of 10^4 iterates, while for $(\alpha - \alpha_*)/\alpha_* \sim 0.1$ we have $\langle\tau\rangle > 10^6$. The reason for this observed persistence of long chaotic transients for relatively large values of $(\alpha - \alpha_*)/\alpha_*$ can be found in the theoretical prediction for $\langle\tau\rangle$. According to the theoretical prediction $\langle\tau\rangle^{-1}$ is zero at α_* (as it must be), and all its derivatives, $d^n\langle\tau\rangle^{-1}/d^n\alpha$, are also zero at $\alpha = \alpha_*$. Thus $\langle\tau\rangle^{-1}$ increases *very slowly* from zero as $(\alpha - \alpha_*)/\alpha_*$ increases. These results contrast with other types of chaotic transients which arise when a strange attractor collides with a smooth (not fractal) basin boundary. In these cases a typical result is $\langle\tau\rangle^{-1} \sim (\alpha - \alpha_*)^{1/2}$ (cf. Ref. 3), and long decay times only exist for $(\alpha - \alpha_*)/\alpha_*$ quite small.

The phenomenology we have described here implies that an experimenter may have to carry on the experiment for a long time in order to distinguish a long-lived chaotic transient from a real chaotic attractor. On the other hand, in experiments where long transients can be seen over a relatively large range of the parameter, unstable-unstable pair annihilation would be a likely cause.

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[4]See, for example, E. Ott, Rev. Mod. Phys. 53, 655 (1981); D. Ruelle, in *Mathematical Problems in Theoretical Physics*, Lecture Notes in Physics Vol. 80 (Springer-Verlag, Berlin, 1978), p. 341.

[5]For the two-dimensional case, in addition to noninvertibility, it also appears that the map must be expanding in some region. More precisely, we expect that the two Lyapunov numbers generated by any orbit on the fractal boundary are both larger than 1 (C. Grebogi, E. Ott, and J. A. Yorke, to be published).

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[8]J. L. Kaplan, J. Mallet-Paret, and J. A. Yorke, to be published.

[9]Numerical and theoretical results verify that the phenomena we observe for Eqs. (3) are not special to this map (cf. Ref. 6).