Permanent Confinement in Four-Dimensional Non-Abelian Lattice Gauge Theory

E. T. Tomboulis
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544
(Received 9 December 1982)

It is shown that the standard SU(2) lattice gauge theory is in a confining phase for all values of the coupling, $0<\beta<\infty$, and $d\leq 4$. The main idea is to show that the electric-flux free energy is bounded by expressions resulting from approximate (Migdal-Kadanoff) renormalization-group transformations.

PACS numbers: 11.15.Ha, 11.10.Gh

There is a substantial body of theoretical evidence indicating that asymptotic freedom and confinement coexist in non-Abelian gauge theories. This is equivalent to saying that the lattice theory is in a confining phase for all values of the coupling, $0<\beta<\infty$. In this Letter I outline an argument establishing this absence of a deconfining transition in the standard (Wilson) SU(2) lattice gauge theory[1] for space-time dimensionality $d \leq 4$.

It is clear that nontrivial estimates on order parameters must incorporate some renormalization group (RG) transformation connecting the short- to the large-distance regime. Although exact RG transformations on higher-dimensional non-Abelian theories are prohibitively difficult, there are several approximate schemes, first introduced by Migdal,[2] which appear to reproduce the phase diagrams of gauge theories rather successfully. The strategy of the following proof will be to bound an appropriate order parameter, the electric-flux free energy, by expressions obtained from such approximate RG procedures. In particular, I will use the Migdal-Kadanoff (MK) "potential moving" decimation scheme,[3] which is already known to produce upper bounds on partition functions. The goal, of course, will be to bound expectations rather than partition functions, a generally considerably harder task. An incomplete, preliminary version of such an argument appeared in an earlier paper.[4] A detailed treatment will be given later.[5]

I consider the standard[1] SU(2) lattice gauge theory defined in terms of bond variables $U[b] \in \mathrm{SU}(2)$ on a finite hypercubic lattice $\Lambda \subset Z^d$, with periodic boundary conditions in all directions. The partition function is[6]

$$Z_\Lambda = \int \prod_{b\in\Lambda} dU[b] \exp A_\Lambda, \tag{1}$$

$$A_\Lambda = \sum_{p\in\Lambda} A_p, \quad A_p = \beta \operatorname{tr} U_p, \quad U_p = \prod_{b\in\partial p} U[b]. \tag{2}$$

Let $\langle\cdots\rangle_0$, $\langle\cdots\rangle$ denote the expectations with measure $\prod_{b\in\Lambda} dU[b]$, and the full measure (1), respectively. The total length of Λ in direction μ will be denoted by A_μ, $\mu=1,\ldots,d$. I typically consider Λ whose "width" $A \equiv A_1 A_2$ in the "transverse" directions [12] is less than their "length" $L \equiv \prod_{\mu\neq 1,2} A_\mu$ in the "longitudinal" directions $(3,\ldots,d)$. Consider a set of plaquettes S which winds once through every [12] plane in Λ, and forms a closed set on the dual lattice. The electric-flux free energy is defined by[7]

$$\exp(-F^{\mathrm{el}}) = \langle \tfrac{1}{2}[1-\prod_{p\in S}\exp(-2A_p)]\rangle$$
$$\equiv \tfrac{1}{2} Z_\Lambda^{-1}[Z_\Lambda - Z_\Lambda^-]. \tag{3}$$

The action in Z_Λ^- is $A_\Lambda^- = A_{\Lambda\setminus S} - A_S$. Equation (3) describes the introduction of color electric flux in the [12] directions in Λ. It is the Z_2-Fourier transform of Z_Λ^-/Z_Λ, the magnetic-flux free energy, which introduces Z_2-magnetic flux on S. It is important to note that (3) is translationally invariant since it does not depend on the choice of S which can be moved around by a change of variables. A consequence of this is an alternative expression for (3):

$$\exp(-F^{\mathrm{el}}) = \tfrac{1}{2}\langle \tfrac{1}{2}[1-\prod_{p\in S}\exp(-2A_p)][1-\prod_{\substack{p\in S'\\ S'\neq S}}\exp(-2A_p)]\rangle. \tag{4}$$

The possible phases of the theory can be completely characterized[7] by (3).

We will be interested in "real-space" RG transformations that relate the theory defined on lattice Λ, with spacing a, and variables U, to the effective theory obtained on lattice Λn, spacing $\lambda^n a$ (integer $\lambda \geq 2$, integer $n \geq 1$), and variables U'. I will take the U' to be simply the old U's along the edges of hypercubes of side length $\lambda^n a$, i.e., the bonds of Λn. I write

$$Z_\Lambda = \exp[F_\Lambda(\lambda^n)] \int \prod_{b\in\Lambda n} dU'[b] \exp[\mathcal{A}_{\Lambda n}(U')] \equiv \exp[F_\Lambda(\lambda^n)]\tilde{Z}_{\Lambda n}. \quad (5)$$

By gauge invariance $\mathcal{A}_{\Lambda n}(U')$ can only depend on traces of (arbitrarily complicated) loops formed out of the U''s. In (5) I explicitly separate the constant piece of the effective action which is common to Z_Λ and Z_Λ^-. At this point we note an important special feature of (3). The flux on S affects only nontrivial representations under the center of the group. This implies that the common constant contribution $F_\Lambda(\lambda^n, \beta)$ contains all constants proportional to $|\Lambda|$. Possible differing contributions can come only from integration over closed *and* topologically nontrivial regions winding around the lattice, and will be included in $\mathcal{A}_{\Lambda n}(U')$. I write $\mathcal{A}_{\Lambda n}(U') = \mathcal{A}_0 + \hat{\mathcal{A}}_{\Lambda n}(U')$, with

$$\mathcal{A}_0 \geq 0, \quad \int \prod_{b\in\Lambda n} dU'[b]\,\hat{\mathcal{A}}_{\Lambda n}(U') = 0. \quad (6)$$

Hence from (3) and (5)

$$\exp(-F^{\text{el}}) = Z_\Lambda^{-1} \exp[F_\Lambda(\lambda^n, \beta)]\tfrac{1}{2}\{\int \prod_{b\in\Lambda n} dU'[b]\exp[\mathcal{A}_{\Lambda n}(U')] - \int \prod_{b\in\Lambda n} dU'[b]\exp[\mathcal{A}_{\Lambda n}^-(U')]\}$$

$$\equiv \tilde{Z}_{\Lambda n}^{-1}\tfrac{1}{2}(\tilde{Z}_{\Lambda n} - \tilde{Z}_{\Lambda n}^-). \quad (7)$$

Consider a decimation from Λ to lattice Λ_λ of spacing λa. An approximate RG transformation can be formulated by adding to the action A_Λ a "decimation operator" Δ_Λ:

$$\Delta_\Lambda = \sum_{\sigma\in\Lambda} \Delta_\sigma, \quad \Delta_\sigma \equiv \sum_{p\in\sigma} a_p A_p, \quad (8)$$

where σ denotes cells of side length λa. The constants a_p satisfy[3]

$$\sum_{p\in\sigma} a_p = 0, \quad (9)$$

and are chosen so that they effectively remove some plaquette interactions while increasing the strength of others at different locations in every σ. Defining

$$Z_\Lambda[\xi] = \int \prod_{b\in\Lambda} dU[b]\exp(A_\Lambda[\xi]), \quad (10)$$

$$A_\Lambda[\xi] = A_\Lambda + \xi\Delta_\Lambda \equiv \sum_{p\in\Lambda} \beta_p(\xi)\,\mathrm{tr}U_p, \quad (11)$$

$$\beta_p(\xi) = \beta(1 + \xi a_p) \geq 0, \quad (12)$$

we can extrapolate between the exact theory ($\xi = 0$), and one application ($\xi = 1$) of a MK decimation $a \to \lambda a$. $Z_\Lambda[\xi]$ is an increasing function of ξ.[3] Similarly, we define $Z_\Lambda^-[\xi]$, with

$$A_\Lambda^-[\xi] = A_\Lambda^- + \xi\Delta_\Lambda^-,$$

$$\Delta_\Lambda^- \equiv \sum_{\sigma\in\Lambda}\sum_{p\in\sigma} (-1)^{S_p} a_p A_p, \quad (13)$$

where $S_p \equiv 1$ if $p\in S$, 0 otherwise. $Z_\Lambda^-[\xi]$ is also an increasing function of ξ.

I now show that

$$d(Z_\Lambda[\xi] - Z_\Lambda^-[\xi])/d\xi \geq 0. \quad (14)$$

This vanishes at $\xi = 0$, since, by translational invariance and (9), the two terms on the left-hand side vanish separately. Therefore, we have to show that

$$d^2(Z_\Lambda[\xi] - Z_\Lambda^-[\xi])/d\xi^2$$
$$= \langle \Delta_\Lambda^2 \exp(A_\Lambda[\xi]) - [\Delta_\Lambda^-]^2 \exp(A_\Lambda^-[\xi])\rangle_0 \geq 0. \quad (15)$$

The proof makes repeated use of reflection positivity (RP).[8] (1) satisfies the fundamental property of RP both about $(d-1)$-dimensional planes containing sites and about planes without sites.[8] It is important to realize that the electric-flux free energy (3) also possesses RP properties. Introducing the factor $1 - \prod_{p\in S}\exp(-2A_p)$ in (1), as is done in (3), results in a measure which, though not positive, is still reflection positive about planes without sites bisecting S. Furthermore, using (4), it is easy to show that we also have RP about planes containing sites. To sketch the main idea behind the proof of (15), let us concentrate on decimations along the "longitudinal" directions. Pick a plane π without sites which divides Λ into two halves Λ_+, Λ_- and also bisects, S, and write $\Delta_\Lambda = \Delta_{\Lambda_+} + \Lambda_{\Lambda_-} + \Delta_S$, $\Delta_\Lambda^- = \Delta_{\Lambda_+} + \Delta_{\Lambda_-} - \Delta_S$. Substituting in (15), we get a cross term $\Delta_{\Lambda_+}\Delta_{\Lambda_-}$ which is positive by RP. The cross terms $\Delta_S\Delta_{\Lambda_+}$, $\Delta_S\Delta_{\Lambda_-}$ can be decomposed into sums of terms. All of these, using our freedom to move the position of S in $A_\Lambda^-[\xi]$ by a change of variables, can then be shown to be positive by RP either about planes without sites, or, using (4), about planes with sites. The terms coming from $\Delta_{\Lambda_+}^2$, $\Delta_{\Lambda_-}^2$ are both of the same form as (15) itself but with the operator Δ_Λ^2 restricted to "half" the lattice. By a change of variables one may now move S to the center of this "half"—recall that our periodic lattice is a circle in every direction—, choose a plane bisecting the new position of S, and repeat the process till the whole lattice is exhausted. One is left with one term Δ_S^2 from each step of this iteration. These terms can also be shown to be positive.[5] Alternatively, it suffices to observe here that for each such term, $A_1 \times A_2$ others were shown to be positive. Hence, in the physically interesting limit of large Λ, they can be ignored. Decimations along the "transverse" directions, i.e., perpendicular to S, can be treated in a similar fashion.

We now note that the proof of (15) can be formulated without any explicit reference to Δ_Λ. To do this, e.g., in the case of decimations in the longitudinal directions, write $\Delta_\Lambda = \sum_{i\in\Lambda} \Delta_{R_i}$, where Δ_{R_i} is the sum of Δ_σ's in a "column" R_i, one lattice spacing wide in the transverse directions. Introduce different parameters ξ_i for each R_i. We have $Z = (Z_\Lambda[\{\xi_i\}])_{\xi_i=\xi}$,

$$\frac{d}{d\xi} Z_\Lambda[\xi] = \sum_{i\in\Lambda} \frac{d}{d\xi_i}(Z_\Lambda[\{\xi_i\}])_{\xi_i=\xi}, \text{ etc.}$$

All splittings that occur in the proof of (15) can now be expressed as splittings in the sum of derivatives $(\sum_{i\in\Lambda} d/d\xi_i)$, and RP of individual terms is restored when at the end we set $\xi_i = \xi$. We also note that, since $\exp(F_\Lambda[\lambda^n]) > 0$, it is clear that $\tilde{Z}_{\Lambda n}$ and $\tilde{Z}_{\Lambda n} - \tilde{Z}_{\Lambda n}^-$ also define RP measures. These observations enable one to apply, with some slight modifications,[5] the argument given for (15), to show that

$$d^2(\tilde{Z}_{\Lambda n}[\xi] - \tilde{Z}_{\Lambda n}^-[\xi])/d\xi^2 \geq 0. \tag{16}$$

Since the first derivatives of $\tilde{Z}_{\Lambda n}[\xi] = \tilde{Z}_{\Lambda n}[\{\beta_p(\xi)\}_\Lambda]$ and $\tilde{Z}_{\Lambda n}^-[\xi] = \tilde{Z}_{\Lambda n}^-[\{\beta_p(\xi)\}_\Lambda]$ again vanish at $\xi = 0$, we obtain

$$d(\tilde{Z}_{\Lambda n}[\xi] - \tilde{Z}_{\Lambda n}^-[\xi])/d\xi \geq 0. \tag{17}$$

We recall that $\tilde{Z}_{\Lambda n}[1]$, $\tilde{Z}_{\Lambda n}^-[1]$ denote the quantities obtained by one complete MK decimation ($\xi = 1$) $a \to \lambda a$, and subsequent exact integration $\lambda a \to \lambda^n a$. It is a well-known feature of the MK transformation that the single plaquette interaction form is maintained, albeit no longer restricted to the fundamental representation. RP is also maintained. Therefore, we may apply the preceding development to $Z_\Lambda[1] \equiv Z_\Lambda^{(1)}$, $Z_\Lambda^-[1] \equiv Z_\Lambda^{-(1)}$; i.e., consider $Z_\Lambda^{(1)}[\xi]$, extract $\tilde{Z}_{\Lambda n}^{(1)}[\xi]$, etc. Iterating m times, we obtain

$$\tilde{Z}_{\Lambda n} - \tilde{Z}_{\Lambda n}^- \leq \tilde{Z}_{\Lambda n}^{(1)} - \tilde{Z}_{\Lambda n}^{-(1)} \leq \cdots \leq \tilde{Z}_{\Lambda n}^{(m)} - \tilde{Z}_{\Lambda n}^{-(m)}. \tag{18}$$

(18) and (7) now give the bound

$$\exp(-F^{\text{el}}) \leq \frac{1}{2} \frac{\tilde{Z}_{\Lambda n}^{(m)}}{\tilde{Z}_{\Lambda n}} \frac{\tilde{Z}_{\Lambda n}^{(m)} - \tilde{Z}_{\Lambda n}^{-(m)}}{\tilde{Z}_{\Lambda n}^{(m)}}, \quad m \leq n. \tag{19}$$

From (6), and Jensen's inequality, we immediately obtain

$$\tilde{Z}_{\Lambda n} \geq 1. \tag{20}$$

All the remaining factors in (19) involve effective actions that evolved after m successive MK decimations. The results of such a transformation are well known.[2,3,9] Starting with the bare plaquette functions at large β, there is, for $d \leq 4$, a continuous evolution from typical weak-coupling behavior, consistent with asymptotic freedom, to the strong-coupling regime. We will take $m = n$ sufficiently large, so that the strong-coupling regime is reached for arbitrary initial large β. Rigorous estimates can be given[5] on the asymptotic form of the plaquette functions in the large n limit. Using these, and known results,[2,3,9] one finds

$$\tilde{Z}_{\Lambda n}^{(n)} \leq \exp\{(\text{const})\exp[-K(\beta,\lambda)\lambda^{2n}]\,|\Lambda n|\}, \tag{21}$$

$$\frac{\tilde{Z}_{\Lambda n}^{(n)} - \tilde{Z}_{\Lambda n}^{-(n)}}{\tilde{Z}_{\Lambda n}^{(n)}} \simeq \frac{L}{\lambda^{(d-2)n}} \exp[-K(\beta,\lambda)\lambda^{2n}], \tag{22}$$

where, for $d = 4$ and small bare coupling $g_0^2 = 2/\beta$,

$$K(\beta,\lambda) = k(\beta)\exp[-b_0^{-1} g_0^{-2}] \tag{23}$$

with $k(\beta) \geq c\beta^a$, for some constants c, a, and

$$b_0(\lambda) = \tfrac{1}{24}(\lambda^2 - 1)/\lambda^2 \ln\lambda, \quad \text{integer } \lambda \geq 2. \tag{24}$$

The transverse size A of Λ must be large enough to allow the necessary n steps to be performed. We may, of course, take n, the number of decimations performed, to actually exhaust the transverse size of Λ; i.e., we set[10] $(\lambda^n)^2 \simeq A$. Then $|\Lambda n| \simeq L/\lambda^{n(d-2)}$, and

$$\exp(-F^{\text{el}}) \leq \exp\{(\text{const})\frac{L}{A^{(d-2)/2}}\exp[-K(\beta)A]\}\frac{1}{A^{(d-2)/2}}L\exp\{-K(\beta)A\}. \tag{25}$$

(25) shows that $\exp(-F^{\text{el}}) \to 0$ exponentially in the transverse size A, as $|\Lambda|$ is allowed to grow, with $L \gg A$, in any power-law fashion. As has been discussed extensively in the literature,[7] this is the signal of the confinement phase. Note that, with (24), the string tension (23) is indeed a lower bound on

the exact asymptotic freedom result.[11] In fact, the bound (25) may be verified "experimentally" for all β by numerical comparison of the MK string tension for $\lambda \geq 2$ with the corresponding Monte Carlo data.[12] At $d=5$, the argument fails to produce (25), since successive MK transformations hit a fixed point, and the exponential falloff of the strong-coupling regime is never reached, no matter how large we take $n(A)$.

The upper bound (25), though sufficient for establishing permanent confinement in the lattice theory, does not allow direct passage to the continuum limit. To demonstrate nonvanishing string tension in the continuum, a sharper estimate, one which converges to the exact value of the b_0 coefficient of the weak-coupling β function, is presumably needed.

The same development could be applied to the two-dimensional chiral $SU(2) \otimes SU(2)$ spin models. Analogous considerations, although with attending technical complications, should extend to general $SU(N)$.

This work was supported in part by National Science Foundation Grant No. PHY80-19754. The author is the recipient of an Alfred P. Sloan Foundation fellowship.

[1]K. G. Wilson, Phys. Rev. D 10, 2445 (1974); J. B. Kogut and L. Susskind, Phys. Rev. D 11, 395 (1975); K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) 110, 440 (1978).

[2]A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 810 (1975) [Sov. Phys. JETP 42, 413 (1976)].

[3]L. P. Kadanoff, Ann. Phys. (N.Y.) 100, 359 (1976).

[4]E. Tomboulis, Phys. Rev. D 25, 606 (1982).

[5]E. Tomboulis, to be published.

[6]In general, A_x will denote the action on a set x.

[7]G. 't Hooft, Nucl. Phys. B153, 141 (1979); G. Mack and V. B. Petkova, Ann. Phys. (N.Y.) 125, 117 (1980); A. Ukawa, P. Windey, and A. H. Guth, Phys. Rev. D 21, 1013 (1980); L. G. Yaffe, Phys. Rev. D 21, 1574 (1980); G. Münster, Nucl. Phys. B180 [FS2], 23 (1981).

[8]J. Fröhlich, R. Israel, E. G. Lieb, and B. Simon, Commun. Math. Phys. 62, 1 (1978); Osterwalder and Seiler, Ref. 1; D. Brydges, J. Fröhlich, and E. Seiler, Ann. Phys. (N.Y.) 121, 217 (1979).

[9]S. Caracciolo and P. Menotti, Nucl. Phys. B180, 428 (1981); M. Nauenberg and D. Toussaint, Nucl. Phys. B190, 217 (1981).

[10]More generally, we may set $A=\lambda^{2N}$, with $N-n=kn+c$ for constant $k, c \geq 0$.

[11]Indeed, the exact $b_0 = \frac{1}{24}(11/\pi^2) > b_0$ $(\lambda \geq 2)$.

[12]M. Creutz, Phys. Rev. D 21, 2308 (1980).

Experimental Study of Stimulated Radiative Corrections on an Atomic Rydberg State

S. Liberman, J. Pinard, and A. Taleb

Laboratoire Aimé Cotton, Centre National de la Recherche Scientifique II, F-91405 Orsay Cédex, France

(Received 4 October 1982)

Light shifts of a Rb Rydberg level, induced by an intense nonresonant electromagnetic field, have been measured for the first time. Agreement between experimental results and theory is satisfactory.

PACS numbers: 32.80.-t, 31.30.Jv

Light shifts in atomic spectra have been extensively studied. Many experiments have been performed for the case in which the frequency of the light is close to the frequency of an atomic resonance. A few calculations have been made to evaluate the influence of the blackbody radiation on the position of Rydberg energy levels.[1,2] This type of calculation generally takes into account the influence of all atomic energy levels including the continuum. Consequently, it requires knowledge of all of the oscillator strengths involved, which can be a serious difficulty. Similar difficulties appear in calculating the effect of an intense resonant electromagnetic (em) field, of strength F and frequency $\omega/2\pi$, on the position of a well-defined atomic energy level E_e. The corresponding shift ΔE_e is given by the general formula

$$\Delta E_e = \tfrac{1}{2} F^2 \sum_f \mu_{ef}{}^2 \left[\frac{1}{E_e - E_f - \hbar\omega} + \frac{1}{E_e - E_f + \hbar\omega}\right]. \qquad (1)$$

However, for the case where the frequency satisfies

$$E_e - E_f \ll \hbar\omega, \qquad (2)$$