Chaos in a Nonlinear Driven Oscillator with Exact Solution

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A nonlinear oscillator externally driven by an impulsive periodic force is investigated. An exact analytical expression is obtained for the stoboscopic or Poincaré map for all values of parameters. The model displays period-doubling sequences and chaotic behavior. The convergence rate of these cascades is in very good agreement with Feingenbaum theory.

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The existence of period-doubling bifurcation and chaos in nonlinear externally driven oscillators has been studied by many authors. A lot of models involving both additive excitations[1] and parametrical ones[2] were numerically investigated with the restrictions imposed by the numerical integration of the equations of motion. As a consequence the results do not present the precision that is habitual in simulation involving unidimensional and multidimensional discrete mappings.[3] In this Letter we present a model of a forced nonlinear oscillator which allows us to obtain an exact analytical expression for the stroboscopic map[4] (analog to the Poincaré surface of section for autonomous systems). To our knowledge this is the first model that is not piecewise linear in which such exact derivation has been performed. By making use of the mentioned map we are able to investigate the stability of periodic solutions as a function of parameters and to calculate the convergence rate of the cascade of period-doubling bifurcations which leads to chaos in some regions of the parameter space.

Consider the equation

$$\ddot{x}+\dot{x}(4bx^2-2a)+b^2x^5-2abx^3+(\omega_0^{\,2}+a^2)x=V_E\omega_E\,\delta(\cos\omega_E t)=V_E\textstyle\sum_n \delta(t-n\tau_E) \tag{1}$$

which can be thought of as modeling an electronic oscillator tuned with nonlinear elements and having its frequency synchronized by means of an external impulsive periodic signal. In (1) $\omega_0=2\pi/\tau_0$ is the proper frequency of the system, a and b are constants, and $\omega_E=2\pi/\tau_E$ and V_E are the frequency and amplitude of the driving pulses, respectively. If $V_E=0$, (1) has the general solution[5]

$$x(t)=\cos(\omega_0t+\varphi)\left\{Ae^{-2at}+\frac{b}{2a}\left[\frac{1}{1+a^2/\omega_0^{\,2}}\right]\left\{1+\frac{a}{\omega_0}\cos(\omega_0t+\varphi)\left(\frac{2a}{\omega_0}\cos(\omega_0t+\varphi)+2\sin(\omega_0t+\varphi)\right)\right.\right.$$
$$\left.\left.-e^{-2at}\left[1+\frac{a}{\omega_0}\cos\varphi\left(\frac{2a}{\omega_0}\cos\varphi+2\sin\varphi\right)\right]\right\}\right\}^{-1/2}, \tag{2}$$

where A and φ are integration constants which are expressed as

$$A=\omega_0^{\,2}[x_0\omega_0+(\omega_0a-x_0^{\,2}-\dot{x}_0)^2]^{-1},$$
$$\sin\varphi=A^{1/2}\omega_0^{\,-1}[x_0a-x_0^{\,3}b-\dot{x}_0],\quad \cos\varphi=x_0\omega_0[x_0\omega_0^{\,2}+(\omega_0a-x_0^{\,2}-\dot{x}_0)^2]^{-1/2}, \tag{3}$$

where $x_0=x(0)$ and $\dot{x}_0=\dot{x}(0)$ are the initial conditions. Thus we have for the autonomous system the following two parametrical equations as solution:

$$x(t)=f(x_0,\dot{x}_0,t),\quad \dot{x}(t)=g(x_0,\dot{x}_0,t), \tag{4}$$

where f and g are given in terms of elementary functions by replacing (3) in (2) and its derivative. When $V_E\neq0$ the only effect of the external force is to produce a discontinuity in the first derivative of $x(t)$. The height of this jump is V_E. Therefore the solution of (1) in the driven case is

$$x_F(t)=\sum_{n=0}^{\infty} f(x_F(n\tau_E),\dot{x}(n\tau_E)+V_E,t-n\tau_E)H(t-n\tau_E)H((n+1)\tau_E-t),$$
$$\dot{x}_F(t)=\sum_{n=0}^{\infty} g(x_F(n\tau_E),\dot{x}(n\tau_E)+V_E,t-n\tau_E)H(t-n\tau_E)H((n+1)\tau_E-t), \tag{5}$$

where H is the unit step function. From (5) it is straightforward to obtain the stroboscopic map by sampling the trajectory at regular time intervals coincident with τ_E. Then the map can be written

$$x_F((n+1)\tau_E)=f(x_F(n\tau_E),\dot{x}_F(n\tau_E)+V_E,\tau_E),$$
$$\dot{x}_F((n+1)\tau_E)=g(x_F(n\tau_E),\dot{x}_F(n\tau_E)+V_E,\tau_E). \quad (6)$$

With the aid of this map we develop a numerical investigation. The first interesting result is the subharmonic entrainment spectrum shown in Fig. 1. There the zones in the (V_E, τ_E) plane are drawn where the stable output of the oscillator is periodic with period commensurable with τ_E. At this regime of dissipation this spectrum is invariant under the shift $\tau_E \to \tau_E + n\tau_0$. Each zone which touches the τ_E axis at the value τ_j is characterized by the rational number $\tau_j/\tau_0 = p/q$ (p and $q \in N$ are, respectively, primes).[6] q is equal to the periodicity of the response and p is the number of oscillations that the system would perform, if the external force vanished, in a time interval τ_j. The sizes of the various entrainment regions

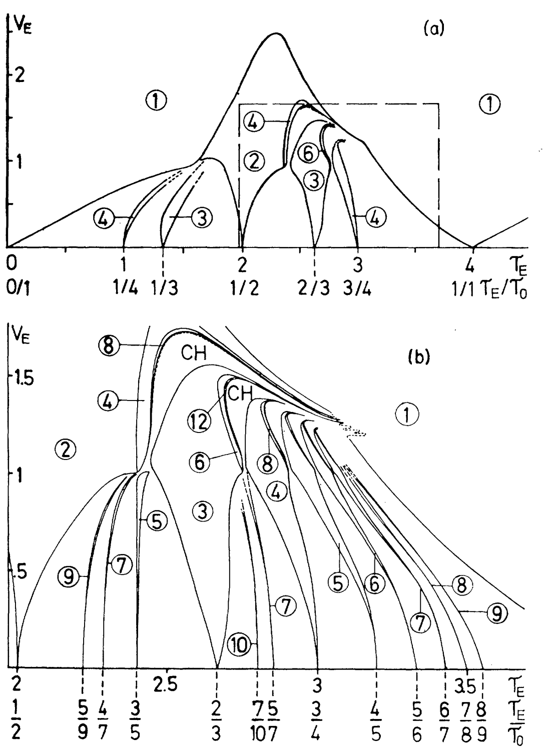

FIG. 1. (a) Stability zones for periodic solutions in the plane (τ_E, V_E) with $a = 1.57079$, $b = 15.7079$, $\omega = 1.57079$. (b) Blowup of the rectangle marked in (a). Numbers inside circles indicate the periodicity which is stable in the corresponding zone. Broken lines indicate lack of good resolution. CH label regions of chaotic behavior.

are ordered in a way related to a concept from number theory, Farey sequences. An n-Farey sequence F_n is the increasing succession of rational numbers whose denominators are less than or equal to n.[7] We call two rational numbers "adjacent" if they are consecutive in F_n for any n. A necessary and sufficient condition for p_1/q_1 and p_2/q_2 to be adjacent is $|p_1q_2 - p_2q_1| = 1$.

A rational number p'/q' belonging to the open interval $(p_1/q_1, p_2/q_2)$ where p_1/q_1 and p_2/q_2 are adjacent will be called "mediant" if there is no other rational in the interval having smaller denominator. It is known that $p'/q' = (p_1+p_2)/(q_1+q_2)$ and is unique. The observation of Fig. 1 and other enlargements not shown here leads us to the following conjecture: The synchronization zone characterized by a mediant number of two adjacent rationals is the greatest of all the zones situated in between those characterized by them. In addition it has, obviously, the least period.

Figure 1(b) shows that the more important entrainment regions have a similar form. They resemble cornucopias, each with the tip attached to the horizontal axis and the other end converging to a point inside of the perfect entrainment region. It is remarkable that all these entrainment regions are of width increasing with V_E when it is less than about 1.0 and above this value each of them decreases in width and folds on itself surrounding a region [marked CH in Fig. 1(b)] which presents a period-doubling route to chaos.[8,9]

Having found such a behavior, we test the Feingenbaum universality. We make use of the fact that, because of the high dissipation, the Jacobian of the stroboscopic map is very small, which ensures that the map neighbors a unidimensional map of the limit cycle on itself. Thus we search for the values of the parameters at which the "critical" point of this "unidimensional" map is periodic.[10] In Table I the values of τ_E (with $V_E = 1.625$) are displayed at which $\partial x_{n+q}/\partial x_n$ evaluated at a stable q-periodic point vanishes, for all $q = 2^r$ with r from 1 to 10. We have named them τ_r. The next two columns list the $\delta_r = |\tau_r - \tau_{r-1}| / |\tau_{r+1} - \tau_r|$ and $\delta_r' = |\delta_r - \delta_{r-1}| / |\delta_{r+1} - \delta_r|$. Since δ' is the convergence rate of δ_r, it allows us to extrapolate the value of the $\lim_{r\to\infty} \delta_r = \delta$. The result is shown in the last column. Notice that it agrees with the exact value to five decimal digits and this accuracy has been achieved without refined numerical techniques.

For most of the points inside the regions CH the response of the oscillator becomes chaotic.

TABLE I. Values obtained through calculations of the convergence rate of the period-doubling sequence at $V_E = 1.625$ and the other values of parameters as in Fig. 1 but with $10.6 < \tau_E < 10.8$. Compare with the exact value of the Feigenbaum constant $\delta = 4.6692016...$.

r	τ_r	δ_r	δ_r'	δ_{extrap}
1	10.743 617 135 97			
2	10.703 113 442 83	3.310 197		
3	10.690 877 398 97	4.231 186	2.825	4.735 833 1
4	10.687 985 528 98	4.557 198	3.731	4.676 595 5
5	10.687 350 957 63	4.644 584	4.535	4.669 306 1
6	10.687 214 331 59	4.663 855	4.586	4.669 243 2
7	10.687 185 036 92	4.668 057 3	4.658	4.669 179 6
8	10.687 178 761 369	4.668 958 9	4.855	4.669 195 6
9	10.687 177 417 267	4.669 144 6		
10	10.687 177 129 398			

However, there also exist regions contained in CH for which the response is entrained. We have found periodicities other than 2^r. The occurrence of these periodicities ensures, as is known,[11,12] the occurrence of chaos for iterated unidimensional maps.

Finally we stress the compatibility of the geometry of the stability regions shown in Fig. 1 with the idea that period-doubling bifurcations are related to the overlap of synchronization horns in the parameter space. It is in fact remarkable that the beginnings of all period-doubling chains in Fig. 1(b) are situated in the neighborhoods of the points at which prolongations of the boundaries of two adjacent regions of stability would intersect. The conjecture of this connection has been made already in Ref. 13 in a different context (see also Glass and Perez[10]).

As a conclusion we want to stress the adequacy of the model presented here as a numerical laboratory for continuous dynamical system. For example, it is easy to check any universal constant (as has been done with δ) as well as the behavior of the power spectrum[14] and statistical properties of the chaotic regime. Details of this calculation will be given elsewhere.

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