Thermodynamics of the Massive Thirring Model: The Discontinuity in Soliton Mass

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A new formulation is presented for the thermodynamics of the massive Thirring model in the attractive-coupling regime. A certain controversy between previous theories is resolved and an overall understanding of the thermodynamics is reached. In particular, it is shown that the soliton mass is discontinuous with the coupling constant at finite temperatures.

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The thermodynamics of the massive Thirring model (MTM) has recently attracted a great deal of theoretical interest.[1-4] Fowler and Zotos[1] first investigated the thermodynamic properties of the MTM for integral coupling constants $P_0=\nu_1$, using the method and formalism developed by Takahashi and Suzuki[5] for the *XXZ* and *XYZ* spin chains. Later, Zotos[2] extended this approach to study the MTM thermodynamics for general coupling constants $P_0=\nu_1+1/\nu_2$ with ν_1 and ν_2 being integers greater than 2. Imada, Hida, and Ishikawa[3] also used the same approach to study the MTM thermodynamics for coupling constants slightly away from the rational points, i.e., $P_0=\nu_1+0$. A controversy seems to exist between these studies in that basic equations obtained in Refs. 1 and 3 are slightly different, although it was proved by Araki[6] that the free energy of this system is an analytic function of the coupling constant. This controversy can be resolved, once an overall understanding of the MTM thermodynamics is reached, as will be demonstrated below. In this paper, we present a new formulation of the MTM thermodynamics for general coupling constants $P_0=\nu_1+1/\nu_2$. The approach is based on the Bethe *Ansatz* (BA) solution of the MTM found by Bergknoff and Thacker (BT),[7] and Korepin,[8] and the thermodynamic analysis given by Yang and Yang.[9] An application of this approach to the weak sine-Gordon (SG) coupling limit ($P_0=\infty$) was reported previously by one of us (S.G.C.).[4] (Note that the quantum SG is equivalent to the charge-neutral sector of the MTM as shown by Coleman.[10]) The basic equations obtained in this approach can be shown to be equivalent to those of Zotos,[2] after some mathematical manipulations. Thus the validity of both theories is confirmed. However, the form of our basic equations is more illustrative in that it is written in terms of physically meaningful quantities, i.e., the renormalized phase shifts for the breather-breather, breather-soliton, and soliton-soliton scatterings. The analysis of our equations shows that the basic equations of Refs. 1 and 3 will lead to the same free energy and breather masses, but different soliton mass at finite temperatures. In other words, at finite temperatures the free energy and breather masses are continuous, whereas the soliton mass is discontinuous at the rational coupling constants $P_0=\nu_1$!

The first step in the present BA approach is to find out allowed fundamental excitations in the system and to quantize a general multiple fundamental excitation by imposing periodic boundary conditions (PBC's) on each excitation in addition to PBC's on the Dirac sea. The fundamental excitations of the MTM were thoroughly studied by BT[7] and Korepin.[8] For the present case $P_0=\nu_1+1/\nu_2$,[2,5] there are ν_1-1 kinds of breathers, ν_2 kinds of Korepin excitations (hereafter K excitations), as well as holes in the Dirac sea. The procedures used here are quite similar to those described in a previous paper[11] for treating breathers. We find that the PBC's on breathers, holes, and K excitations can all be written in a BA form of quantizing the associated physical momenta, in terms of the renormalized phase shifts, $\Delta_{ij}{}^{bb}$ (ith breather $-$ jth breather), $\Delta_i{}^{bh}$ (ith breather $-$ hole), where $i,j=1,2,\ldots,\ \nu_1-1$, Δ^{hh} (hole $-$ hole), $\Delta_m{}^{\mathrm{K}h}$ (mth K excitation $-$ hole), and $\Delta_{mn}{}^{\mathrm{KK}}$ (mth K excitation $-$ nth K excitation), where $m,n=0,1,\ldots,\ \nu_2-1$.[12] The renormalized phase shifts between breathers and K excitations are found to be identically zero. The result of quantizing these fundamental excitations can be written in a compact BA form,

$$P_j(\alpha_k{}^j)=(2\pi/L)I_k{}^j+L^{-1}\textstyle\sum_i \Delta_{ji}*\rho_i, \qquad (1)$$

where L is the system size, $I_k{}^j$ is an integer, and $\rho_i(\alpha)$ is the density of distribution for the ith excitation in the rapidity α space (i and j run over breathers, holes, and K excitations). We have introduced a convenient notation $a * b \equiv \int_{-\infty}^{\infty} d\alpha' a(\alpha' - \alpha) b(\alpha')$. The $P_j(\alpha)$ is the physical momentum of the jth excitation; it equals $M_s \sinh(\gamma\alpha)$ for holes, $2M_s \sin[(\frac{1}{2}j\pi)(2\gamma - 1)] \times \sinh(\gamma\alpha)$ for breathers, and zero for K excitations, where[13] $M_s = \mu/(\pi - \mu)$ is the zero-temperature soliton mass in the unit of the zero-temperature free soliton mass $M_s{}^0$ (for $P_0 = 2$), $\mu = (P_0 - 1)\pi/P_0$, and $\gamma = \pi/2\mu$.

After rewriting the PBC's on the bare excitation of the MTM in a BA form (1), we then apply the method of Yang and Yang[9] for the BA thermodynamics. The basic procedures are formally the same as those reported in Ref. 4, except that here we also include the K excitations. Subtracting from Eq. (1) the same equation with $\alpha_k{}^j$ replaced by $\alpha_{k-1}{}^j$ gives

$$dP_j(\alpha)/d\alpha = 2\pi\,\mathrm{sgn}(j)[\rho_j(\alpha) + \tilde{\rho}_j(\alpha)] + \sum_i (\partial/\partial\alpha)\Delta_{ji} * \rho_i, \tag{2}$$

where $\mathrm{sgn}(j) = 1$ for j denoting breathers, hole, and the 0th K excitation, and -1 for j denoting other K excitations, and $\tilde{\rho}_j(\alpha)$ is the density of "omitted α value" distribution.[9] The factor $\mathrm{sgn}(j)$ comes from the fact that the bare masses are negative for K excitations with $j = 1, 2, \ldots, \nu_2 - 1$. This equation provides a relation between densities $\rho_j(\alpha)$ and $\tilde{\rho}_j(\alpha)$, since $dP_j(\alpha)/d\alpha$ are known functions. The internal energy, entropy, and hence the free energy of the system can also be expressed as functionals of densities $\rho_j(\alpha)$ and $\tilde{\rho}_j(\alpha)$ in the same manner as described in Eqs. (1) and (2) of Ref. 4. With use of Eq. (2) the free energy can be expressed as functionals of ρ_i's alone. Minimizing the free energy with respect to independent variations of densities $\delta\rho_i$ gives [with $\tilde{\rho}_j(\alpha)/\rho_j(\alpha) \equiv \exp(\epsilon_j(\alpha)/T)$]

$$\epsilon_j(\alpha) = E_j(\alpha) + (T/2\pi)\sum_i \mathrm{sgn}(i)(\partial/\partial\alpha)\Delta_{ij} * \ln[1 + \exp(-\epsilon_i/T)], \tag{3}$$

where the Boltzmann constant is set equal to 1, the temperature is measured in the unit of the zero-temperature free soliton mass, $M_s{}^0$, and $E_j(\alpha) = (1/\gamma)[dP_j(\alpha)/d\alpha]$ is the physical energy of fundamental excitations.

Solving the coupled integral equations (2) and (3) we can find densities of fundamental excitations, the free energy, and hence all the thermodynamic quantities. Equation (3) can be shown to be equivalent to the basic equations of Ref. 2, after a long manipulation. Thus the validity of both theories is confirmed. However, we wish to comment that our equations are more illustrative than those of Ref. 2 in that the physical meaning of the quantities $\Delta_{ij}{}^{bb}$, $\Delta_i{}^{bh}$, and Δ^{hh} as the renormalized phase shifts is unambiguous, as they can be obtained independently from the S-matrix factorization theory.[14] Two things are worth mentioning before we proceed further. First, a compact expression for the free-energy density in the rapidity α space can be derived from Eqs. (2) and (3),

$$F(\alpha) = -(\gamma T/2\pi)\sum_i E_i(\alpha)\ln\{1 + \exp[-\epsilon_i(\alpha)/T]\}. \tag{4}$$

Note that the summand in the right-hand side of Eq. (4) is just the free energy of a free fermion with the energy $\epsilon_i(\alpha)$. Furthermore, we can show that the inclusion of K excitations does not alter the previous proof[4] that the quantity $\epsilon_i(\alpha)$ represents the excitation spectrum at $T \neq 0$.[15] Thus it is seen from Eq. (4) that the effect of the phase-shift interaction between fundamental excitations appears in the free energy only through the thermal renormalization of hole and breather energies. Second, it is noted that the hole contribution in the free energy can alternatively be represented as the soliton-plus-antisoliton contribution. It is evident by symmetry that the soliton and antisoliton densities are equal to each other; that is, the charge neutrality is established at any point in the rapidity space. Therefore, we can replace holes and K excitations by solitons and antisolitons at any point in the rapidity space. Also note that K excitations do not carry free energies, but carry charges. Denoting the soliton (antisoliton) energy as $\epsilon^s(\alpha)$ we can switch the representation from the hole to soliton-antisoliton picture by the relation [cf. Eq. (4)]

$$1 + \exp(-\epsilon^h/T) = [1 + \exp(-\epsilon^s/T)]^2. \tag{5}$$

We now consider the limit $\nu_2 \to \infty$, where the integral equation for $\epsilon_n{}^K(\alpha)$ in Eqs. (3) becomes an algebraic equation (with $\eta \equiv \epsilon/T$):

$$\eta_n{}^K = \ln[1 + \exp(-\eta^h)] + \sum_{m<n} 2m\ln[1 + \exp(-\eta_m{}^K)] + (2n-1)\ln[1 + \exp(-\eta_n{}^K)] + \sum_{m>n}^{\infty} 2n\ln[1 + \exp(-\eta_m{}^K)]. \tag{6}$$

The solution to Eq. (6) can be found through a similar procedure as given by Johnson.[16] We obtain

$$[1+\exp(\eta_n{}^K)]^{1/2}=n+[1+\exp(-\eta^h)]^{1/2} \quad \text{for } n\geqslant 1. \tag{7}$$

With this solution, we can eliminate $\eta_n{}^K$ in Eqs. (3). Thus we have obtained coupled integral equations for $\eta^h(\alpha)$ and $\eta_j{}^b(\alpha)$ $(j=1,2,\ldots,\ \nu_1-1)$ for the case $P_0=\nu_1+0$. Let us first consider the simplest case $\nu_1=2$, i.e., the free MTM limit. In this case the integral equations for $\eta^h(\alpha)$ and $\eta_1{}^b(\alpha)$ become simple algebraic equations,

$$\begin{aligned}\eta_1{}^b&=E_1{}^b/T+\ln[1+\exp(-\eta_1{}^b)]+\ln[1+\exp(-\eta^h)],\\ \eta^h&=E^h/T+\ln[1+\exp(-\eta_1{}^b)]-\ln\{1+1/[1+\exp(-\eta^h)]^{1/2}\}\end{aligned} \tag{8}$$

with solution

$$[1+\exp(-\eta^h)]^{1/2}=(Z+2)/(Z+1), \quad 1+\exp(-\eta_1{}^b)=(Z+1)^2/(Z^2+2Z), \tag{9}$$

where $Z\equiv\exp[E^h(\alpha)/T]$. Substituting η^h and $\eta_1{}^b$ given by Eq. (9) into Eq. (4) gives

$$F(\alpha)=-(\gamma T/\pi)\cosh(\gamma\alpha)\ln\{1+\exp[-E^h(\alpha)/T]\}, \tag{10}$$

which is precisely the free energy of the free MTM. Note that this demonstration of the continuity of the free energy is nontrivial. There is a certain difference between the cases $P_0=\nu_1$ and ν_1+0. What happens when P_0 passes the point $P_0=\nu_1$ from larger values is that the highest breather dissociates into a soliton-antisoliton pair.[7] That is, some frozen degrees of freedom of the soliton and antisoliton motion, due to the existence of the highest breather, are released when P_0 passes ν_1 from larger values. Since the free energy is continuous at $P_0=2$,[6] the soliton mass should decrease suddenly at this point to cover a sudden disappearance of parts of the free energy carried by the dissociated breathers. Switching from the hole picture to soliton-antisoliton picture by Eq. (5), we find the magnitude of discontinuity in the soliton mass at $P_0=2$ as

$$\Delta M_s(2,\ T)=\ln[1+\exp(-1/T)]. \tag{11}$$

The above argument for the case $P_0=2$ applies to the general case $P_0=\nu_1$ as well. Returning to the basic equation for $\eta_j{}^b$ and η^h for the case $P_0=\nu_1+0$, we can first show that

$$[1+\exp(\eta_{\nu_1-1}{}^b)]^{1/2}=\exp(\eta^h)\{1+[1+\exp(-\eta^h)]^{1/2}\}. \tag{12}$$

Substituting $\eta_{\nu_1-1}{}^b$ back into the original equation and introducing a new variable $\bar\eta^s(\alpha)=\bar\epsilon^s(\alpha)/T$ by

$$[1+\exp(-\eta^s)][1+\exp(-\eta_{\nu_1-1}{}^b)]=1+\exp(-\bar\eta^s), \tag{13}$$

we arrive at coupled integral equations for $\eta_j{}^b$ $(j=1,2,\ldots,\ \nu_1-2)$ and $\bar\eta^s$ which are precisely those of Ref. 1 for the case $P_0=\nu_1$.[12] This means that the free energy and breather masses are continuous at $P_0=\nu_1$, but at finite temperatures the soliton mass is discontinuous, i.e., $\epsilon^s(0)\neq\bar\epsilon^s(0)$, because of the dissociation of the highest breather into a soliton-antisoliton pair.

Finally, we have solved the basic equations numerically for the cases $P_0=\nu_1+1/\nu_2$, ν_1+0, and ν_1 by the iteration method, and calculated the coupling-constant dependences of the free energy, lowest-breather mass, and soliton mass. Figure 1 shows these quantities at $T=2$. Note that both the free energy and lowest-breather mass change continuously at $P_0=\nu_1$, but the soliton mass suffers a sudden change at these points. It is also clear in Fig. 1 that the magnitude of discontinuity in the soliton mass $\Delta M_s(\nu_1,\ T)$ becomes smaller for larger ν_1, as is expected. As for the temperature dependence of ΔM_s [cf. Eq. (11) for the case $\nu_1=2$], numerical results show that it always increases with temperature, but the temperature dependence becomes weaker for larger ν_1.

In summary, we have presented a new formulation of the MTM thermodynamics for the case $P_0=\nu_1+1/\nu_2$ $(\nu_1,\ \nu_2\geqslant 2)$ which almost covers the entire attractive-coupling regime. Taking the limit $\nu_2\to\infty$ in the basic equation obtained, we have proved that the free energies obtained for the cases $P_0=\nu_1$ and $P_0=\nu_1+0$ are the same, in consistency with Araki's theorem.[6] Breather masses are also found to be continuous at $P_0=\nu_1$. On the other hand, we find that at finite temperatures the soliton mass suddenly decreases when P_0 passes ν_1 from larger values. The discontinuity of the soliton mass has its origin in a fundamen-

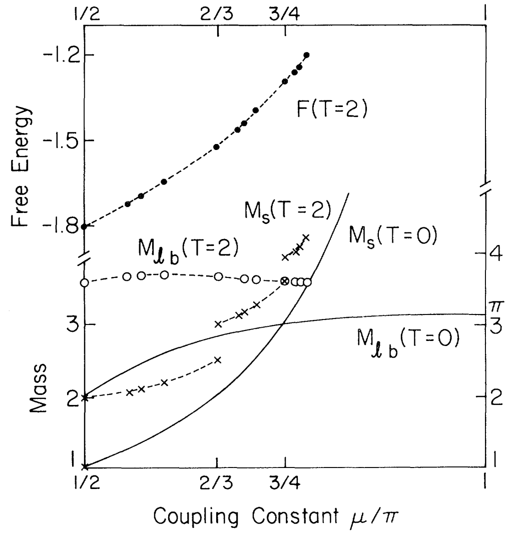

FIG. 1. Plot of the free energy F (solid circles), soliton mass M_s (crosses), and lowest-breather mass M_{1b} (open circles) as functions of the coupling constant μ/π at $T=2$. These discrete points are connected by dashed curves for clarity. The soliton and lowest-breather masses at $T=0$ are also plotted (solid curves) for comparison. The mass and temperature are measured in the unit of the zero-temperature free-soliton mass, M_s^0.

tal effect in the soliton system: the sharing of energy and degrees of freedom among various types of extended objects. This effect was also a key point to the understanding of the breather problem.[4,17] Since this effect is more pronounced for higher temperatures, the magnitude of discontinuity in the soliton mass should increase with temperature.

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