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Self-Triality of the Ashkin-Teller Model

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It is shown that the Ashkin-Teller model has self-triality, which is a generalization of Kramers-Wannier self-duality proposed some time ago and illustrated with examples. In the Hamiltonian formalism, it means that the original Hamiltonian may be reexpressed in terms of either of two disorder variables without change of form and that the relation between all three variables is fully symmetric.

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The Ashkin-Teller model (ATM)[1] involves a two-dimensional lattice, at the sites of which reside "spins" s and t which can take the values ± 1. The spins interact with their nearest neighbors in a manner to be made precise later. The goal is to calculate Z, the partition function, and explore the thermodynamics.

Now, just as the Ising model approximates a magnetic system of vector magnetic moments with scalar spins that can take values ± 1, the ATM can be used to model a system with two magnetic moments per site. We could also see it as a lattice-gas problem in which the lattice sites are occupied by four species of atoms, the type at any site being given by the joint values of s and t there.

Why study such a simplified model in two dimensions? The answer is that two-dimensional systems are now experimentally accessible and that according to the modern theory of critical phenomena, a given realistic Hamiltonian and a crude approximation to it can be equivalent as far as critical phenomena are concerned.

For the Ising model, Kramers and Wannier[2] established the property of self-duality. This implies first of all that $Z(T)=Z(T^*)$, where the dual temperature $T^*(T)$ falls from ∞ to 0 as T rises from 0 to ∞. Thus if there is a unique critical temperature T_c, it must be the solution to $T^*(T_c)=T_c$. Nowadays we know that more is involved in self-duality: Not only is $Z(T)=Z(T^*)$, but we can view the highly disordered high-temperature system as a highly ordered system in a dual set of variables. (The key ideas will be reviewed here, while further details can be found in Ref. 3.)

In Ref. 4 I showed that self-duality, with all its attendent implications, has a natural extension called self-triality and provided two concrete examples. Here I will show that the ATM possesses self-triality. We begin with a quick review of the Ising model, self-duality, and self-triality.

Self-duality in the Ising model.—Consider the partition function for an anisotropic Ising model,

$$Z=\sum_{s_i}\exp\left(\sum_i K_x s_i s_{i+x}+K_t s_i s_{i+t}\right), \qquad (1)$$

where $s_i=+1$ or -1 are Ising spins located at the sites i of a two-dimensional lattice, coupled to their neighbors at sites $i+x$ and $i+t$ displaced in the "x" and "t" directions, respectively. It is possible to reduce the evaluation of Z to the eigenvalue problem of a transfer matrix T.[5] If we consider, following Fradkin and Susskind,[6] the case

$$K_x=\lambda_x\tau, \quad \exp(-2K_t)=\lambda_t\tau, \qquad (2)$$

where $\tau \to 0$, T has the form $T = 1 - \tau H$, where

$$-H = \sum_{n=-\infty}^{\infty} [\lambda_x \sigma_3(n)\sigma_3(n+1) + \lambda_t \sigma_1(n)]. \tag{3}$$

Here $\sigma_i(n)$ are the Pauli matrices obeying the following algebra:

$$\{\sigma_i(n), \sigma_j(n)\} = 2\delta_{ij}, \quad [\sigma_i(n), \sigma_j(m)] = 0 \text{ for } m \neq n. \tag{4}$$

The free energy is simply related to the energy of the ground state $|0\rangle$ while $\langle 0|\sigma_3|0\rangle = \langle\sigma_3\rangle$ is the magnetization.[5]

Since the physics depends only on the ratio $\lambda = \lambda_x/\lambda_t$, let us set $\lambda_x + \lambda_t = 1$. When $\lambda_x = 1$, $|0\rangle = |\pm\rangle$, which are states with all spins up or down. The order parameter $\langle\sigma_3\rangle = \pm 1$ in this case, which corresponds to zero temperature. The λ_t part of H flips the spins and disrupts this order. Beyond some critical value λ_c, $\langle\sigma_3\rangle$ vanishes. Self-duality gives us a way to find λ_c as follows. Let us define a new set of variables

$$\mu_3(n) = \prod_{-\infty}^{n} \sigma_1(m), \quad \mu_1(n) = \sigma_3(n)\sigma_3(n+1) \tag{5}$$

which obey the same algebra as the σ's. In terms of these

$$-H = \sum_n [\lambda_x \mu_1(n) + \lambda_t \mu_3(n-1)\mu_3(n)]. \tag{6}$$

Since $H(\mu)$ has the same form as $H(\sigma)$, and σ and μ are isomorphic, the spectrum, and in particular, the ground-state energy E_0, are invariant under $\lambda_x \leftrightarrow \lambda_t \to 1/\lambda$. Thus a singularity in $E_0(\lambda)$ at any value of λ implies the same at its inverse. Assuming the phase transition is unique, we see that it must be at $\lambda = 1$ or $\lambda_x = \lambda_t = \frac{1}{2}$.

One calls μ_3 the disorder variable with respect to σ_3 for the following reason. Consider the vacuum state $|+\rangle$ with all spins up. When $\mu_3 = \prod_{-\infty}^{n} \sigma_1$ acts on it, it flips all the spins from $-\infty$ up to n. Such a state, which connects two different vacua, is called a kink or a soliton. While a single kink is forbidden in the ordered state by the imposed boundary conditions, a pair of them is allowed and will produce an island of spins that are oriented opposite to the macroscopic magnetization. In this sense, kinks disorder the system. For $\lambda_t \gg \frac{1}{2}$ or $\lambda \gg 1$, we see from Eq. (6) that the system that seems disordered in σ_3 is ordered in the disorder variable μ_3. Of course the names order and disorder are relative, and one could start with Eq. (6) and think of μ_3 as the order parameter. [The formulas for σ as a function of μ will have exactly the same form as Eq. (5).] For the order-disorder concept in the two-dimensional version, see Refs. 7 and 8.

In summary, the salient features of self-duality are the following:

(i) There exists a nonlocal change of variables that preserves the form of H up to a permutation of the coupling constants. (ii) The new variables are isomorphic to the old, i.e., obey the same algebra. (iii) The new variables can be interpreted as disorder variables with respect to the old and vice versa. The transformation laws have the same form when we go from the old to the new variables as when we revert back.

Self-triality.—It is now natural to use the term self-triality for the case wherein: (i) There exist two sets of isomorphic disorder variables and H has the same form as before when expressed in terms of either. (ii) There exists complete symmetry between all three variables, and in particular, the formulas for the change of variables have the same form as their inverses.

A spin model based on Dirac (instead of Pauli) matrices and a quantum field theory [the O(8) Gross-Neveu model[9]] exhibiting the self-triality are discussed in Ref. 4.

Self-triality of the Ashkin-Teller model.—Here each site contains two Ising spins s_i and t_i and in the same notation as before,

$$Z = \sum_{s_i, t_i} \exp\{\sum_i [J_x^{(1)} s_i s_{i+x} + J_x^{(2)} t_i t_{i+x} + J_x^{(3)} s_i t_i s_{i+x} t_{i+x} + (x \to t)]\}. \tag{7}$$

Consider the case

$$J_x^{(i)} = \tau\lambda_x^{(i)} \ (i = 1, 2, 3); \ \exp[-2(J_t^{(1)} + J_t^{(2)})] = \tau\lambda_t^{(3)} \text{ and cyclic permutations} \tag{8}$$

with $\tau \to 0$. We get as before

$$-H = \sum_n [\lambda_t^{(2)} \sigma_1(n) + \lambda_t^{(1)} \tau_1(n) + \lambda_t^{(3)} \sigma_1(n)\tau_1(n) + \lambda_x^{(1)} \sigma_3(n)\sigma_3(n+1) + \lambda_x^{(2)} \tau_3(n)\tau_3(n+1) + \lambda_x^{(3)} \sigma_3(n)\sigma_3(n+1)\tau_3(n)\tau_3(n+1)], \tag{9}$$

where τ are the Pauli matrices associated with the spins t_i. They obey the same algebra as the σ's and commute with them. Following Wegner[10] and Wu and Wang,[11] let us perform a duality transformation on the τ matrices alone as follows:

$$\mu_1(n)=\tau_3(n)\tau_3(n+1), \quad \mu_3(n)=\prod_{n+1}^{\infty}\tau_1(m)\equiv\overrightarrow{\prod}\tau_1, \tag{10}$$

and get H as a function of σ and μ. Further let us restrict ourselves to the following manifold in parameter space:

$$\lambda_t^{(i)}=\lambda_x^{(i)}=\omega_i, \quad i=1,2,3 \tag{11}$$

which is invariant under $\sigma \leftrightarrow \mu$. In this subspace,

$$-H=\sum_n\{\omega_1[\sigma_3(n)\sigma_3(n+1)+\mu_3(n-1)\mu_3(n)]+\omega_2[\sigma_1(n)+\mu_1(n)]$$
$$+\omega_3[\mu_1(n)\sigma_3(n)\sigma_3(n+1)+\sigma_1(n)\mu_3(n-1)\mu_3(n)]\}. \tag{12}$$

(The slight asymmetry between σ and μ is due to the fact that strictly speaking, they are defined on two different lattices dual to each other and which must be exchanged along with σ and μ.)

Let us normalize H in the following symmetric way:

$$\omega_1+\omega_2+\omega_3=1. \tag{13}$$

It is convenient to imagine an equilateral triangle of unit height and to associate with each interior point the coordinates ω_1, ω_2, and ω_3 corresponding to the perpendicular distances to the three sides. [Equation (13) will be automatically satisfied.]

The natural order parameters for H in Eq. (12) are clearly σ_3 and μ_3. When $\omega_1=1$, that is, at one of the corners of the triangle, $|\langle\sigma_3\rangle|=|\langle\mu_3\rangle|=1$. As we turn on ω_2 and ω_3 at the expense of ω_1, this order will be reduced and eventually wiped out. To explore the behavior of the system beyond this domain, we introduce two sets of disorder variables (to tackle regions with dominant ω_2 or ω_3). The first set is

$$\gamma_1(n)=\sigma_3(n)\sigma_3(n+1), \quad \gamma_3(n)=\prod_{n+1}^{\infty}\sigma_1(m)\equiv\overrightarrow{\prod}\sigma_1, \quad \tau_1(n)=\mu_3(n-1)\mu_3(n), \quad \tau_3(n)=\prod_{-\infty}^{n-1}\mu_1(m)\equiv\overleftarrow{\prod}\mu_1. \tag{14}$$

It can be verified that γ and τ obey the same algebra as σ and μ and that the inverse transformations have the same form. [The τ we see here is the same one we began with, see Eq. (10).] In terms of these

$$-H=\sum_n\{\omega_1[\gamma_1(n)+\tau_1(n)]+\omega_2[\tau_3(n)\tau_3(n+1)+\gamma_3(n-1)\gamma_3(n)]$$
$$+\omega_3[\gamma_1(n)\tau_3(n)\tau_3(n+1)+\tau_1(n)\gamma_3(n-1)\gamma_3(n)]\}. \tag{15}$$

We now see that when $\omega_2=1$, the system is ordered in τ_3 and γ_3, with the ω_1 and ω_3 terms disrupting this order.

The other set of disorder variables is

$$\eta_1(n)=\sigma_1(n), \quad \eta_3(n)=(\overleftarrow{\prod}\mu_1)\sigma_3(n), \quad \xi_1(n)=\mu_1(n), \quad \xi_3(n)=\mu_3(n)\overrightarrow{\prod}\sigma_1, \tag{16}$$

in terms of which

$$-H=\sum_n\{\omega_1[\eta_3(n)\eta_3(n+1)\xi_1(n)+\xi_3(n-1)\xi_3(n)\eta_1(n)]$$
$$+\omega_2[\eta_1(n)+\xi_1(n)]+\omega_3[\xi_3(n-1)\xi_3(n)+\eta_3(n)\eta_3(n+1)]\}. \tag{17}$$

Now we see that near $\omega_3=1$, the system is ordered in η_3 and ξ_3 while the ω_1 and ω_2 terms disrupt the order.

It is easy to see that instead of starting with H in Eq. (12) and identifying σ_3 and μ_3 as order parameters with respect to which (γ_3, τ_3) and (ξ_3, η_3) are disorder parameters, we could just as well start with H in Eqs. (15) or (17) and exchange the roles of the variables. There is complete symmetry between the variables at every stage.

While the natural variables for the present analysis are (σ_3, μ_3) and their disorder variables, we can still ask how things look in terms of the original variables σ_3, τ_3, and $\sigma_3\tau_3$. Near $\omega_1=1$, the system is ordered in σ_3 and in $\mu_3=\overrightarrow{\prod}\tau_1$, which disorders τ_3 and $\sigma_3\tau_3$. Near $\omega_2=1$, the order is in τ_3 and $\gamma_3=\overrightarrow{\prod}\sigma_1$, which disorders σ_3 and $\sigma_3\tau_3$. Finally near $\omega_3=1$, the order is in $\eta_3=(\overleftarrow{\prod}\mu_1)\sigma_3=\tau_3\sigma_3$ and in ξ_3

$=\mu_3\overrightarrow{\prod}\sigma_1 = \overrightarrow{\prod}(\sigma_1\tau_1)$ which disorders σ_3 and τ_3 but not $\sigma_3\tau_3$. This picture does not correspond to triality as defined above since σ_3, τ_3, and $\sigma_3\tau_3$ are not disorder variables with respect to each other. Hence our initial choice of variables.

Self-triality will of course tell us a lot about the phase diagram in the triangle introduced earlier. I do not wish to pursue this aspect here since the phase structure and symmetries (besides triality) are extensively discussed in the literature,[12-14] some even in the Hamiltonian form.[15]

Let us note that while duality in the Ising model relates high and low temperatures, here it relates different phases in a plane that is left invariant under such high-low transformation.

Finally we relate our H to a fermion field theory. Let us introduce four Majorana (Hermitian) operators,

$$\psi_1(n) = (\overleftarrow{\prod}\sigma_1)(\prod_{-\infty}^{\infty}\tau_1)\sigma_2(n), \quad \psi_2(n) = (\overleftarrow{\prod}\sigma_1)(\prod_{-\infty}^{\infty}\tau_1)\sigma_3(n), \quad \chi_1(n) = (\overleftarrow{\prod}\tau_1)\tau_2(n), \quad \chi_2(n) = (\overleftarrow{\prod}\tau_1)\tau_3(n), \tag{18}$$

obeying

$$\{\psi_i(n), \psi_j(m)\} = 2\delta_{ij}\delta_{mn}, \quad \{\chi_i(n), \chi_j(m)\} = 2\delta_{ij}\delta_{mn}, \quad \{\psi_i(n), \chi_j(m)\} = 0. \tag{19}$$

In terms of these

$$-H = \sum_n \{\omega_1[i\psi_1(n)\psi_2(n+1) - i\chi_1(n)\chi_2(n)] + \omega_2[i\chi_1(n)\chi_2(n+1) - i\psi_1(n)\psi_2(n)] + \omega_3[\psi_1(n)\psi_2(n+1)\chi_1(n)\chi_2(n+1) + \psi_1(n)\psi_2(n)\chi_1(n)\chi_2(n)]\} \tag{20}$$

and we can think of self-triality as a property of this field theory.

To conclude, it is shown here that the Ashkin-Teller model too exhibits self-triality, as defined here and in Ref. 4. While the present analysis was in the Hamiltonian form, the phenomenon undoubtedly has a counterpart in the classical two-dimensional version of the problem. Given the close link between Baxter's eight-vertex model[16] and this one[10,17] we can expect self-triality there as well.

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