Quantum Conduction on a Cayley Tree

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Quantum resistance of a Cayley tree composed of one-dimensional random scatterers is considered. The strength of a scatterer is characterized by its typical resistance ρ_0. It is shown that a metal-insulator transition occurs at some critical scattering strength ρ_{0c}, which depends on the coordination number of the tree. The resistance of an infinite tree diverges for $\rho_0 > \rho_{0c}$ and saturates at some finite value for $\rho_0 < \rho_{0c}$.

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The main feature of the electron energy spectrum in disordered systems is the existence of a mobility edge E_c, which separates extended and localized states. At low temperatures the electronic transport in such systems crucially depends on the position of the Fermi level E_F. For $E_F \gg E_c$ (weak scattering, or classical transport regime) an electron wave packet diffuses essentially as a classical particle. On the other hand, for E_F close to E_c quantum interference effects are strong; they lead to long-range correlations in the system and, for $E_F < E_c$, to a complete localization of the electron eigenstates. When E_F approaches E_c from above, the zero-temperature dc conductivity approaches zero as

$$\sigma(E_F) \sim (E_F - E_c)^t, \tag{1}$$

which defines the conductivity exponent t. Scaling arguments, supplemented by perturbation calculations in the weak-scattering regime,[1-3] led to a conclusion that this metal-insulator transition (the Anderson transition) occurs only at dimensionality $d > 2$ (for $d \leq 2$ all the eigenstates in a disordered system are localized). For $d = 2 + \epsilon$, with $\epsilon \to 0$, the conductivity exponent $t \to 1$.[4,5]

The purpose of this Letter is to study the Anderson transition on a Cayley tree (Bethe lattice), which is a branching graph without loops (Fig. 1). Such a graph is believed to represent a space of infinite dimensionality,[6] and thus a transition with "mean field" exponents is expected. The existence of an Anderson transition on a Cayley tree has been established long ago.[7] However, conductivity studies, as far as I know, have been limited to a calculation,[8] based on Kubo's formula, for a tight-binding Anderson model.[9] The authors of Ref. 8 find a minimum metallic conductivity at the transition, in contrast with Eq. (1). Below I develop an analytic approach to the problem and find a continuous metal-insulator transition.

The approach is based on a scattering (S matrix) formalism, which has been successfully used in recent scaling studies of quantum transport.[10-13] In this formalism one consideres a large number of "black boxes"—scatterers—fitted together. A scatterer is supposed to represent a region of the disordered medium and it is characterized by a random S matrix chosen from some statistical distribution. In a scaling theory one combines several scatterers into a single, renormalized, scatterer, and tries to calculate the distribution for the renormalized S matrix. The S matrix of a "box" is then related to the observable transport properties such as the resistance. In particular it is by this approach that Anderson *et al.*[10] have derived the "quantum Ohm's law" for adding one-dimensional resistances, ρ_1 and ρ_2, in series:

$$1 + \rho = (1 + \rho_1)(1 + \rho_2), \tag{2}$$

where ρ is the combined resistance, and all resistances are measured in units $\pi\hbar/e^2$. As explained in Ref. 10, resistances in Eq. (2) are

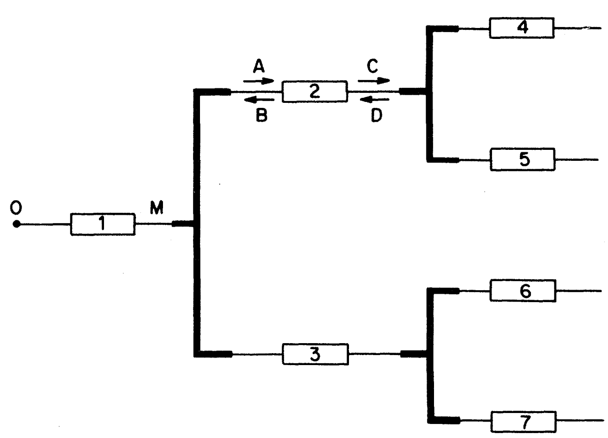

FIG. 1. A Cayley tree of one-dimensional single-channel scatterers (boxes). Each channel can carry two waves propagating in opposite directions. As an example, the waves incoming (A, D) into and outgoing (B, C) from scatterer 2 are shown by arrows. Splitters are shown by heavy lines.

"typical" or "scale" resistances of the corresponding distributions.

The model I consider is shown in Fig. 1. Each box represents a one-dimensional random scatterer, which is characterized by a 2×2 scattering matrix. This matrix relates the two outgoing amplitudes to the two incoming amplitudes. For instance, for scatterer 2

$$\begin{pmatrix} B \\ C \end{pmatrix} = \begin{pmatrix} r_2 & t_2 \\ t_2 & r_2' \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}, \tag{3}$$

where r_2 (r_2') is the reflection amplitude on the left (right) of the scatterer, and t_2 is the transmission amplitude (time-reversal symmetry is assumed). To connect one-dimensional scatterers in parallel I introduce an ideal device—a "splitter", with one channel on the left and q channels on the right. In Fig. 1 splitters are represented by heavy lines. In this figure $q=2$, but in the following I keep q general. A splitter is characterized by a (nonrandom) S matrix

$$S_{\mathrm{sp}} = \begin{pmatrix} 0 & \underline{t} \\ \underline{\tilde{t}} & \underline{r}' \end{pmatrix}. \tag{4}$$

Here zero means that there is no reflection in the single channel on the left. $\underline{t}$ is a row matrix describing transmission from any of the q channels on the right to the channel on the left. Assuming for simplicity the same transmission amplitude for all q channels, one has $\underline{t}=(1/\sqrt{q})(1, 1,\ldots, 1)$, where the factor $1/\sqrt{q}$ insures unitarity. Further, $\underline{\tilde{t}}$ is the transposed (column) matrix describing transmission from left to right. Finally, $\underline{r}'$ is a $q\times q$ matrix describing reflection on the right end of a splitter; namely, the element r_{mn}' is the reflection amplitude from channel n to channel m on the right. Again, assuming the same amplitude for any pair m,n ($m\neq n$) and requiring unitarity, one has $r_{mn}'=\delta_{mn}-(1/q)$.

Let us now consider an array of q arbitrary scatterers (not necessarily the "elementary" scatterers of Fig. 1) connected in parallel via a splitter (Fig. 2). An incident (from the left) wave of a unit amplitude splits into q identical waves. Each of these waves is partially reflected by the corresponding scatterer and partially transmitted to the right. The reflected waves propagate back and are partially reflected back to the right into various channels, etc. Solving this multiple-scattering problem, one can calculate the total reflection amplitude r:

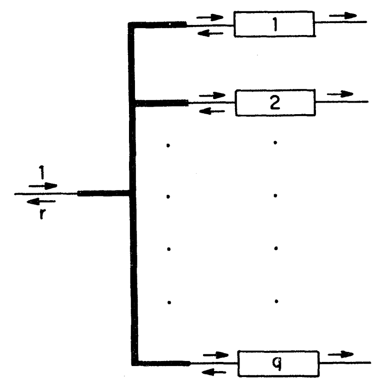

FIG. 2. An array of q scatterers connected via a splitter. 1 and r on left refer to the amplitudes of the incident and reflected waves.

$$\frac{r}{1-r} = \frac{1}{q}\sum_{n=1}^{q}\frac{r}{1-r_n}, \tag{5}$$

where r_n is the reflection amplitude, from the left, of the nth scatterer.

$r_n \equiv a_n \exp(i\theta_n)$ and $r\equiv ae^{i\theta}$ are random complex variables. The problem is to calculate the distribution $F(a,\theta)$ for the variable r from the distributions $f(a_n,\theta)$ for the variables r_n (the same distribution for all r_n is assumed). This is a standard problem in probability theory, which is best solved by writing $r_n/(1-r_n)\equiv u_n+iv_n$, i.e.,

$$u_n(a_n,\theta_n) = a_n\frac{\cos\theta_n - a_n}{1-2a_n\cos\theta_n + a_n^2},$$

$$v_n(a_n,\theta_n) = a_n\frac{\sin\theta_n}{1-2a_n\cos\theta_n + a_n^2}, \tag{6}$$

and calculating the characteristic function

$$p(\alpha,\beta) = \int_0^1 da_n\int_0^{2\pi}d\theta_n f(a_n,\theta_n)\exp\left\{\frac{i}{q}\left[\alpha u_n(a_n,\theta_n)+\beta v_n(a_n,\theta_n)\right]\right\}. \tag{7}$$

The probability distribution for the random variables $u=(1/q)\sum_n u_n$ and $v=(1/q)\sum_n v_n$ is then given by

$$\varphi(u,v) = (1/4\pi^2)\iint d\alpha\, d\beta\exp[-i(\alpha u+\beta v)][p(\alpha,\beta)]^q. \tag{8}$$

Finally, returning to the variables a,θ [related to u,v by Eq. (6), with subscript n omitted], one calculates $F(a,\theta)$.

Assuming random uniformly distributed phases one can write $f(a_n,\theta)=(1/2\pi)\varphi(a_n)$. For $a_n\ll1$, the r_n (and r) terms in denominators of Eq. (5) can be neglected, and the problem reduces to the wellknown problem of a random walk in a plane.[14] I will not discuss this case, but rather consider the opposite case, $1-a_n \equiv \epsilon_n/2 \ll 1$, which is the relevant case for the study of the Anderson transition. The reflection coefficients $R_n \equiv a_n^2$ of the q scatterers in Fig. 2 are, in this case, close to unity and are distributed according to some (narrow) distribution with an average $R = \langle a_n^2 \rangle \approx 1 - \langle \epsilon_n \rangle \equiv 1-\epsilon$. The small parameter ϵ enables one to perform explicitly the outlined calculation [Eqs. (6)–(8)] and to obtain, in powers of ϵ, the (average) reflection coefficient $\tilde{R} = \langle a^2 \rangle \equiv 1 - \tilde{\epsilon}$ of the whole array. To leading order in ϵ the result is simply $\tilde{\epsilon} = q\epsilon + O(\epsilon^2)$. This relation can be rewritten in terms of resistances by means of the Landauer formula[15,16] $\rho = R/(1-R)$, which relates the (dimensionless) resistance ρ of a scatterer to its reflection coefficient R. Since $\epsilon, \tilde{\epsilon} \ll 1$, one has

$$\tilde{\rho} = (\rho/q) + O(1), \tag{9}$$

where ρ is a typical resistance of a scatterer in Fig. 2, while $\tilde{\rho}$ is the resistance of the whole array.

Let us now return to the Cayley tree in Fig. 1. All the elementary resistances 1, 2, 3, etc., are chosen from the same distribution, with a typical resistance ρ_0, and are uncorrelated with each other. I denote by ρ_N the resistance of a tree of N generations (scatterer 1 represents the first generation, scatterers 2, 3 represent the second generation, etc.). ρ_N is defined as the resistance between the initial point 0 and an electrode connecting all the scatterers of Nth generation. Clearly, ρ_N can be viewed as two resistances in series: the first (elementary) resistance ρ_0 and the rest of the tree, starting from point M. But the rest of the tree has a structure of the type shown in Fig. 2, i.e., q resistances in parallel connected via a splitter, where each of these q resistances corresponds to a tree of $(N-1)$ generations. Thus assuming that ρ_N and ρ_{N-1} are large, i.e., the system is close to the metal-insulator transition, and combining Eqs. (2) and (9), one obtains

$$1+\rho_N = (1+\rho_0)[1+(1/q)\rho_{N-1} + O(1)]. \tag{10}$$

It follows from this recursion equation that $\rho_{0c} = q-1$ is the critical value of the parameter ρ_0: For $\rho_0 > \rho_{0c}$, ρ_N increases exponentially with N, when $N \to \infty$ (insulating behavior); for $\rho_0 < \rho_{0c}$, Eq. (10), for $N \to \infty$, has a finite nonzero solution

$$\rho_\infty \sim [1-(1+\rho_0)/q]^{-1} \sim (\rho_{0c} - \rho_0)^{-1}, \tag{11}$$

i.e., (for $q > 1$), there is a metal-insulator transition at ρ_{0c}.

In order to compare the results with those of real lattices, one needs the resistivity, or conductivity, rather than the resistance ρ_∞. A meaningful definition of the conductivity on a Cayley tree is not a trivial question, which, in the context of the classical percolation transition, has been discussed in Ref. 6. There one distinguishes between the "microscopic" conductivity σ_{mic} (proportional to $1/\rho_\infty$) and the physically interesting "macroscopic" conductivity σ_{mac} (proportional to the average current flowing through an elementary resistance in the presence of a unit electric field). Far from the percolation threshold σ_{mic} and σ_{mac} differ just by a numerical factor, while near threshold, because of the complicated nature of a conducting path $\sigma_{\text{mic}} \gg \sigma_{\text{mac}}$. Since no percolationlike phenomena are expected in the present problem (the distribution of the elementary resistances is a well-behaved distribution, with a typical value ρ_0, rather than a binary distribution of the percolation problem), I believe that σ_{mic} and σ_{mac} should be roughly the same. It follows then from Eq. (11) that the conductivity exponent [Eq. (1)] is $t=1$. This is a somewhat surprising result since, as has been mentioned, t approaches unity also near the lower critical dimensionality $d=2$ (the Cayley tree, however, represents an infinite dimensionality).

In summary, an S-matrix approach to the conductivity problem on a Cayley tree has been developed. It is demonstrated that the ("microscopic") conductivity approaches zero when the resistance of an elementary scatterer, ρ_0, approaches the value $q-1$ from below. The corresponding exponent is equal to unity.

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Light Emission from Electron-Injector Structures

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Surface-plasmon-polariton–mediated luminescence is observed when electrons are injected into thin Al films from the conduction band of SiO_2. These electron-injector structures are strikingly similar to light-emitting tunnel junctions, although tunneling can be ruled out as the driving mechanism. The emission arises from the energy relaxation of the steady-state hot-electron distribution which exists in the metal under continuous current injection. The same mechanism must explain much of the luminescence from tunnel junctions.

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Since the seminal papers of Lambe and McCarthy,[1,2] it has been widely accepted that light emission from metal-insulator-metal (MIM) tunnel junctions results from a two-stage process. First, an electron tunnels inelastically, losing its energy to a collective excitation of the junction. Second, in the presence of surface roughness, this excitation may radiate. Since the energy loss occurs in the insulating region of the junction,[3,4] inelastic tunneling should most efficiently excite electromagnetic modes with large energy density in this region. Theoretical attention[4,5] was therefore initially focused on the "slow wave" or junction mode,[6] with fields concentrated between the metal electrodes. However, radiation from electrodes consisting of many small metal balls[7-9] has been shown to be mediated by localized plasmons.[9-11] More recently light emission has been demonstrated via the "fast" surface-plasmon polariton, which has a maximum energy density at the outer electrode surface.[12-15] To explain the efficiency with which the fast mode is excited, Laks and Mills[16] proposed a phenomenological model in which the inelastic-tunneling current fluctuations extend into the metal electrodes on both sides of the insulating junction. However, the excitation efficiency appears to be much higher than predicted by this theory.[14] This and several other puzzling results, discussed at length in Refs. 14 and 17, have suggested the importance of an alternative excitation mechanism, the injection of hot electrons into the metal by *elastic* tunneling.

If this mechanism is correct, it should not matter *how* the electrons are injected, as long as they enter the metal with energies several electronvolts in excess of the Fermi energy. Here we report the observation of surface-plasmon-polariton–mediated light emission when electrons are injected from the conduction band of SiO_2 into