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Uncertainty in Quantum Measurements

David Deutsch
Center for Theoretical Physics, The University of Texas at Austin, Austin, Texas 78712, and Astrophysics Department, Oxford University, Oxford, England
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The object of this Letter is to show that except in the case of canonically conjugate observables, the generalized Heisenberg inequality does not properly express the quantum uncertainty principle. It is, in general, too weak. An inequality is obtained which does express the principle.

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In quantum theory, any single observable or commuting set of observables can in principle be measured with arbitrary accuracy.[1-3] But there is in general an irreducible lower bound on the uncertainty in the result of a simultaneous measurement of noncommuting observables. Equivalently, there is an upper bound on the accuracy with which the values of noncommuting observables can be simultaneously prepared. These are qualitative statements of the uncertainty principle in quantum theory. My purpose here is to obtain a quantitative expression of the principle. We shall see that the customary generalization

$$V_{\hat{A}}(|\psi\rangle)V_{\hat{B}}(|\psi\rangle) \geq \tfrac{1}{4}|\langle\psi|[\hat{A},\hat{B}]|\psi\rangle|^2 \quad (1)$$

of Heisenberg's inequality[4]

$$V_{\hat{x}}(|\psi\rangle)V_{\hat{p}}(|\psi\rangle) \geq \tfrac{1}{4} \quad ([\hat{x},\hat{p}]=i), \quad (2)$$

though it is of course true, will not fit the bill. The quantity

$$V_{\hat{A}}(|\psi\rangle) \equiv \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2 \quad (3)$$

is the variance of $\hat{A}$ in the state $|\psi\rangle$ and the units are chosen so that $\hbar = 1$.

In order to express the principle

$$\begin{bmatrix}\text{Uncertainty in the result of a}\\ \text{measurement of } \hat{A} \text{ and } \hat{B}\end{bmatrix} \geq \begin{bmatrix}\text{An irreducible lower}\\ \text{bound}\end{bmatrix} \quad (4)$$

quantitatively, I shall seek a theorem of linear algebra in the form

$$\mathfrak{U}(\hat{A},\hat{B},|\psi\rangle) \geq \mathfrak{B}(\hat{A},\hat{B}). \quad (5)$$

Here $\hat{A}$ and $\hat{B}$ are the observables which are simultaneously measured or prepared and $|\psi\rangle$ is the relative state representing the outcome of the measurement or preparation.

It is logically possible that the bound $\mathfrak{B}$ could also depend on the initial state of the system, but this could not be the case in quantum theory where there always exists a dynamical evolution which transforms any initial state into any other.

Heisenberg's inequality (2) has the form (5) but its generalization (1) does not. The right-hand side of (1) is not a fixed lower bound but is itself a function of $|\psi\rangle$. For some choices of $|\psi\rangle$, not confined to those representing definite simultaneous values for $\hat{A}$ and $\hat{B}$, it even vanishes. Even if it were not for the other deficiencies which we shall encounter, this alone would disqualify (1) as a full expression of the uncertainty principle (4).

In order to represent a quantitative physical notion of "uncertainty," $\mathcal{U}$ must at least possess the following elementary property: If and only if $|\psi\rangle$ is a simultaneous eigenstate of $\hat{A}$ and $\hat{B}$ (i.e., when both observables possess definite values), $\mathcal{U}$ must take a fixed minimum value, which I shall for the moment assume to be zero. Otherwise $\mathcal{U}$ must exceed zero. From this we can infer a property of $\mathcal{B}$: that it must vanish if and only if $\hat{A}$ and $\hat{B}$ have an eigenstate in common or have eigenstates arbitrarily close together, and $\mathcal{B}$ must exceed zero otherwise. Although Heisenberg's inequality (2) "satisfies" these requirements (vacuously) its generalization (1) does not.

A less obvious but equally necessary requirement on $\mathcal{U}$, which is also not satisfied by (1), may be obtained by considering the case when $\hat{A}$ and $\hat{B}$ have discrete spectra. It is important that (5) be at least as capable of expressing the uncertainty principle for discrete observables (such as spins and angular momenta) as for continuous ones. For although continuous observables such as the position $\hat{x}$ are familiar enough, they are really unphysical idealizations: The set of possible results of any realizable measurement is always countable, since the state space of any apparatus with finite spatial extent has a countable basis. Notice that if $\hat{A}$ is discrete, the only physically significant values of the variance $V_{\hat{A}}(|\psi\rangle)$ are zero and nonzero. For a mere (nondegenerate) relabeling of the eigenvalues of a discrete observable has no physical significance, yet under such a relabeling $V_{\hat{A}}(|\psi\rangle)$ can be multiplied by any desired positive value. Thus the quantitative appearance of (1) is illusory in the case of discrete observables. More generally, arguing along these lines leads to the conclusion that a quantitative measure of uncertainty for a discrete observable $\hat{A}$ must not depend on the eigenvalues of $\hat{A}$ (except possibly for taking into account their degeneracy structure).

Similarly, $\mathcal{U}(\hat{A}, \hat{B}, |\psi\rangle)$ can depend only on $|\psi\rangle$ and the sets $\{|a\rangle\}$ and $\{|b\rangle\}$ of eigenstates of $\hat{A}$ and $\hat{B}$. It follows that $\mathcal{B}(\hat{A}, \hat{B})$ can depend only on the set $\{\langle a|b\rangle\}$ of inner products between these eigenstates.

The most natural measure of the uncertainty in the result of a measurement or preparation of a *single* discrete observable is the "entropy",

$$S_{\hat{A}}(|\psi\rangle) = -\sum_a |\langle a|\psi\rangle|^2 \ln|\langle a|\psi\rangle|^2. \tag{6}$$

$S_{\hat{A}}(|\psi\rangle)/\ln 2$ is precisely the deficiency in the information which the outcome $|\psi\rangle$ gives about further measurements which might be made on $\hat{A}$ (compared with the case when $|\psi\rangle$ is one of the eigenstates $\{|a\rangle\}$). This suggests the choice

$$\mathcal{U}(\hat{A}, \hat{B}, |\psi\rangle) = S_{\hat{A}}(|\psi\rangle) + S_{\hat{B}}(|\psi\rangle). \tag{7}$$

When $\hat{A}$ and $\hat{B}$ are discrete, this satisfies all the requirements placed on $\mathcal{U}$ in the above discussion. When they are continuous, there is only the problem that (7) is then not nonnegative definite. This is no more than a technical problem and, in view of my remarks about the unphysical nature of continuous observables, I shall not pursue it here.

I have not been able to obtain a constructive expression for the bound $\mathcal{B}(\hat{A}, \hat{B})$ corresponding to (7). The nonconstructive definition

$$(\hat{A}, \hat{B}) = \inf_{|\psi\rangle} \{S_{\hat{A}}(|\psi\rangle) + S_{\hat{B}}(|\psi\rangle)\} \tag{8}$$

is of course not especially useful. However, the existence and relevant properties of $\mathcal{B}$ may be established as follows: We have

$$S_{\hat{A}}(|\psi\rangle) + S_{\hat{B}}(|\psi\rangle) = -\sum_a |\langle\psi|a\rangle|^2 \ln|\langle\psi|a\rangle|^2 - \sum_b |\langle\psi|b\rangle|^2 \ln|\langle\psi|b\rangle|^2 \tag{9}$$

$$= -\sum_{ab} |\langle\psi|a\rangle|^2 |\langle\psi|b\rangle|^2 (\ln|\langle\psi|a\rangle|^2 + \ln|\langle\psi|b\rangle|^2). \tag{10}$$

The parenthesized quantity in (10) is nonpositive definite and is never greater than its value when $|\psi\rangle$ lies midway between $|a\rangle$ and $|b\rangle$, i.e., when

$$|\psi\rangle = 2^{-1/2}(1 + |\langle a|b\rangle|)^{-1/2}[|a\rangle + \exp(-i\arg\langle a|b\rangle)|b\rangle]. \tag{11}$$

Thus

$$S_{\hat{A}}(|\psi\rangle) + S_{\hat{B}}(|\psi\rangle) \geq -2\sum_{ab} |\langle\psi|a\rangle|^2 |\langle\psi|b\rangle|^2 \ln[\tfrac{1}{2}(1 + |\langle a|b\rangle|)], \tag{12}$$

which implies

$$S_{\hat{A}}(|\psi\rangle)+S_{\hat{B}}(|\psi\rangle) \geq \mathcal{B}(\hat{A},\hat{B}) \geq 2\ln \frac{2}{1+\sup\{|\langle a|b\rangle|\}}\,. \tag{13}$$

Since the right-hand side of (13) has all the properties requisite of $\mathcal{B}$, it follows that $\mathcal{B}$ does also. Equation (13) is a satisfactory quantitative expression of the uncertainty principle.

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Exact Solution of a Nonlinear Eigenvalue Problem in One Dimension

B. Sriram Shastry
Tata Institute of Fundamental Research, Bombay 400005, India
(Received 22 July 1982)

An exact solution of the eigenvalue problem

$$\{-d^2/dx^2+\Delta[1-a_0\rho(x)]\}\psi_n(x)=E_n\psi_n(x)$$

with $\rho(x)=\sum_{n=1}^{N}|\psi_n(x)|^2$ and with periodic boundary condition is presented. The solution gives rise to a density wave with $[\text{const}-\rho(x)]$ proportional to $\text{sn}^2[(x/\lambda)|m]$ for suitable values of the parameters λ and m. The solution rests upon some remarkable properties of the solutions of Lamé's equation.

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The present work is concerned with an exact solution of a Hartree-like nonlinear eigenvalue problem (the N-component nonlinear Schrödinger equation) in one dimension,

$$\{-d^2/dx^2+\Delta[1-a_0\rho(x)]\}\psi_n(x)=E_n\psi_n(x)\,, \tag{1}$$

where $0\leq x\leq L$, $1\leq n\leq N$, $a_0=L/N$, Δ is a coupling constant,

$$\rho(x)=\sum_{n=1}^{N}|\psi_n(x)|^2 \tag{2}$$

and ψ_n's are orthonormal and obey periodic boundary conditions $\psi_n(x+L)=\psi_n(x)$.

This problem has been considered approximately in various contexts earlier. In the context of the nuclear many-body problem[1] Eqs. (1) appear as Hartree's equations for N spinless fermions constrained to a box of size L, interacting through a two-body attractive δ-function potential with strength Δ. In an important paper entitled "Structure of Nuclear Matter" Overhauser[1] presented a weak-coupling theory ($\Delta\to 0$) and showed that there exist solutions with broken translation invariance, in addition to the trivial solutions $\psi(x)\sim\exp(iKx)$, which are always lower in (total) energy. Overhauser found that the fermion density develops a sinusoidal shape with an amplitude that depends nonanaltyically on the coupling Δ as $\Delta\to 0$. These "charge-density waves" have found wide application in condensed-matter physics. Exact solutions of the equations do not appear to be known for large N, although the time-dependent version of the problem with boundary conditions

$$\psi_n(x)\xrightarrow[x\to\pm\infty]{}0$$

seems to be better understood.[2]

These equations may be viewed as the Hartree-Fock (HF) equations for a system of $2N$ spin-$\frac{1}{2}$ fermions on a ring with an attractive δ-function interaction, which is of course a celebrated exactly solvable many-body problem.[3,4] Lieb and