Two-Dimensional Angular Momentum in the Presence of Long-Range Magnetic Flux

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It is shown that eigenvalues of two-dimensional angular momentum remain integer valued in the magnetic field of a solenoid, contrary to published assertions that they are modified by the flux. For a vortex, flux does contribute, and the angular momentum can fractionize, as asserted in the literature, provided phases of wave functions are chosen consistently with the solenoid problem. Long-range effects of flux, the distinction between orbital and canonical angular momentum, and interactions with Cooper pairs are essential to this argument.

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It is now well established that physical systems containing solitons carry unexpected quantum numbers. For example, in the presence of a monopole—a three-dimensional soliton—a particle with integral spin can form a half-integer–spin bound state[1]; vis. a boson can be converted to a fermion.[2] There has appeared in this Journal a series of papers alleging similar behavior in the presence of a vortex—a two-dimensional soliton—and even in the presence of a conventional solenoid.[3] While we do support the statement about the vortex, we show here that for the solenoid it is false: Angular momentum has conventional eigenvalues. All peculiarities can be explained by the difference between the kinematical orbital angular momentum, and the conserved, canonical angular momentum, but this is a familiar distinction whenever velocity-dependent forces occur, as in the interaction of a charged particle with a magnetic field, which is under discussion here.

The published statements[3] concerning charged-particle–solenoid interactions have already been criticized in the context of a three-dimensional geometry.[4] We amplify this criticism by discussing the problem entirely in a two-dimensional setting. Moreover, we consider time-varying solenoid configurations, since time dependence limits ambiguity in constants of motion. Of course, the two-dimensional rotation group, O(2), being Abelian, does not force a group-theoretical quantization on the angular momentum's eigenvalues. Nevertheless, by adhering to a conventional, Noether definition of the rotation generator, we show that a spinless particle, in the presence of a solenoid, possesses integer angular momentum eigenvalues. (It is to be recalled that in the monopole problem, Noether's theorem correctly gives the full angular momentum, including its nonkinematical part.)

Since the O(2) group does not give a unique definition for the angular momentum in a two-dimensional solenoid field, one may add an arbitrary constant to the angular momentum operator, thus obtaining arbitrary eigenvalues. Equivalently, one may allow an arbitrary angular phase in the wave function. However, we believe that this is neither natural nor required. Moreover, we show that the proposed modification[3] cannot be supported in the general time-dependent situation, and runs counter to the correspondence principle.

Also we examine the vortex, and confirm that the angular momentum can become half-integral,[3] provided that phases are chosen naturally, as in the solenoid problem with integral eigenvalues.

Charged particle in the solenoid field.—We consider a (heavy) charged particle, of mass μ, which moves in a magnetic field B, pointing along the z axis, homogeneous in that direction, and invariant against rotations around that axis. Dynamical motion is confined to the two-dimensional x-y plane, and is governed by the (nonrelativistic) Lagrangian,

$$L=\tfrac{1}{2}\mu\vec{v}^{\,2}+(e/c)\vec{v}\cdot\vec{A}(\vec{r}),\quad \vec{v}=\dot{\vec{r}},\tag{1}$$

where $\vec{r}(t)$ is the particle's coordinate,

$$\vec{r}=\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}r\cos\varphi\\r\sin\varphi\end{pmatrix};\quad \hat{r}=\vec{r}/r.\tag{2}$$

$\vec{A}$ is a vector potential for the magnetic field, which is a two-dimensional (pseudo) scalar: $B=\nabla\times\vec{A}$. We take B regular everywhere, and short range, rapidly approaching zero for $r>R$. Thus the "solenoid" is the region $r<R$.[5]

A convenient form for the vector potential is

$$A^i(\vec{r})=(2\pi)^{-1}\epsilon^{ij}(\hat{r}^j/r)\Phi(r);\qquad A^r=0,\quad A^\varphi=(2\pi r)^{-1}\Phi.\tag{3}$$

The magnetic field is determined by Φ', which is negligible for $r>R$, and vanishes at the origin:

$$B=(2\pi r)^{-1}\Phi'. \qquad (4)$$

We shall chose Φ to vanish at $r=0$ as well; this insures that $\vec{A}$ is nonsingular there, but it is long range, since for $r>R$, Φ will be nonzero, rapidly approaching its asymptote Φ_∞. The significance of Φ is that it is the flux in a circle of radius r,

$$\int_{r'\leq r} d^2r'\, B(\vec{r}')=\Phi(r), \qquad (5)$$

and Φ_∞ is the total flux. Note that $\vec{A}$ is rotationally covariant; no fixed, external vectors occur in its definition.

Equation (3) represents a choice of gauge in which $\vec{A}$'s profiles are symmetric and nonsingular. An alternative choice gives a short-range potential, at the expense of a singularity at the origin. The singular configuration, related to (3) by a singular gauge transformation,

$$\vec{A}_S=\vec{A}-\nabla\Omega_S, \quad \Omega_S=(\varphi/2\pi)\Phi_\infty, \qquad (6)$$

carries the following rotationally covariant profiles:

$$A_S{}^i(\vec{r})=-(2\pi)^{-1}\epsilon^{ij}(\hat{r}^j/r)[\Phi(r)-\Phi_\infty];$$
$$A_S{}^r=0, \quad A_S{}^\varphi=(2\pi r)^{-1}(\Phi-\Phi_\infty). \qquad (7)$$

Here $\vec{A}$ vanishes for $R>r$, but the origin must be excluded, in order that (7) describe the magnetic field (4) (rather than one with an additional δ-function singularity at the origin). A regularization may be achieved by considering a family of nonsingular gauge functions, parametrized by α,

$$\Omega_S{}^\alpha(\vec{r})=(\varphi/2\pi)\Phi_\infty\rho_\alpha(r);$$
$$\rho_\alpha(0)=0, \quad \rho_\alpha(\infty)=1, \qquad (8)$$

which tend to Ω_S as $\alpha\to 0$ ($\rho_\alpha \xrightarrow[\alpha\to 0]{} 1$). Note that the profiles of the regularized gauge potential are no longer rotationally covariant, because there now is a radial component proportional to $\varphi\rho_\alpha{}'$.

Thus, if we are willing to accept a rotationally noncovariant description, a nonsingular, short-range vector potential can be employed. Making this choice, we find it more convenient to use the gauge

$$\vec{A}_{NS}=\vec{A}-\nabla\Omega_{NS}; \quad \Omega_{NS}=(\varphi/2\pi)\Phi \qquad (9)$$

so that

$$A_{NS}{}^i(\vec{r})=-\hat{r}^i(\varphi/2\pi)\Phi'(r);$$
$$A_{NS}{}^r=-(\varphi/2\pi)\Phi', \quad A_{NS}{}^\varphi=0. \qquad (10)$$

Here we shall not analyze in detail the singular, short-range gauge description (6), (7), because the required regularization (8) renders it analogous to the nonsingular, unsymmetric, short-range gauge (9), (10). We shall begin with the symmetric gauge (3). To show that its long-range tail presents no complications, our results will be rederived in the unsymmetric, nonsingular, short-range gauge (9), (10).

An infinitesimal rotation

$$\delta r^i=-\omega\epsilon^{ij}r^j; \quad \delta r=0, \quad \delta\varphi=\omega \qquad (11)$$

leaves the Lagrangian (1) invariant, when the vector potential is given by (3), $\delta L=0$. Therefore, the conserved angular momentum, according to Noether's theorem, is $M=(\delta L/\delta\vec{v})\cdot\delta\vec{r}/\omega$. The canonical momentum $\vec{p}\equiv\delta L/\delta\vec{v}$ differs from the kinematical momentum $\mu\vec{v}$, since there are velocity-dependent (magnetic) forces present: $\vec{p}=\mu\vec{v}+(e/c)\vec{A}$. As a consequence the angular momentum possesses a contribution in addition to the orbital one:

$$M=\vec{r}\times\vec{p}=\vec{r}\times\mu\vec{v}+(e/c)\vec{r}\times\vec{A}$$
$$=\vec{r}\times\mu\vec{v}+(e/2\pi c)\Phi. \qquad (12)$$

In particular, in the region outside the solenoid $r>R$, the orbital angular momentum is supplemented by the total flux $(e/2\pi c)\Phi_\infty$.

Let us notice, moreover, that nothing depends on the flux being constant in time. The angular momentum is conserved as a consequence of rotational invariance. Of course for time-independent flux, $\vec{r}\times\mu\vec{v}$ and Φ_∞ separately are constant outside the solenoid, but this special case obscures the general situation. With time-dependent flux only the combination (12) is a constant of motion.

In quantum mechanics $\vec{p}$, not $\mu\vec{v}$, becomes the operator $(\hbar/i)\nabla$. Hence in the presence and in the absence of the solenoid, the angular momentum operator is the same: $M=(\hbar/i)\vec{r}\times\nabla=(\hbar/i)\partial/\partial\varphi$. Completeness of its eigenfunctions or Hermiticity of the operator requires that the eigenvalues be integer spaced (in units of $\hbar$). However, this does not eliminate the possibility of an arbitrary common phase, proportional to φ, in all the eigenfunctions, which would shift all angular momentum eigenvalues by the same, arbitrary quantity. Such arbitrariness cannot be eliminated by group-theoretical arguments.

In familiar situations, the further requirement of single valuedness (or continuity) is set on the wave function, and thus the remaining φ-dependent phase is removed, leaving integral eigenvalues for M.

For the solenoid, there is no reason to abandon single-valued (or continuous) wave functions: The potential is rotationally symmetric and nonsingular; that it ranges to radial infinity as $1/r$ does not force a modification (see below). Hence we conclude that, contrary to published assertions,[3] the angular momentum spectrum consists of $\hbar$ times the integers m[6]: $M\Psi=\hbar m\Psi$.

In the Schrödinger equation,

$$i\hbar\frac{\partial}{\partial t}\Psi=H\Psi=\frac{1}{2\mu}\left(\vec{p}-\frac{e}{c}\vec{A}\right)^2\Psi, \tag{13}$$

angular variables are separated as usual:

$$\Psi(t,\vec{r})=(2\pi)^{-1/2}e^{im\varphi}u_m(t,r), \tag{14a}$$

$$i\hbar\frac{\partial}{\partial t}u_m=\left(-\frac{\hbar^2}{2\mu}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{[\hbar m-(e/2\pi c)\Phi]^2}{2\mu r^2}\right)u_m. \tag{14b}$$

The centrifugal barrier is shifted by the flux, but the angular momentum remains an integral quantum.

Let us now see how things look in the short-range gauge (9) and (10). The Lagrangian

$$L_{\rm NS}=\tfrac{1}{2}\mu\vec{v}^2-(e/c)\vec{v}\cdot\hat{r}(\varphi/2\pi)\Phi'(r) \tag{15}$$

is no longer invariant under rotations (11); rather it changes by a total time derivative $\delta L_{\rm NS}=\omega\, d[-(e/2\pi c)\Phi]/dt$. Consequently angular momentum is still conserved; its form is

$$M_{\rm NS}=\vec{r}\times\vec{p}+(e/2\pi c)\Phi. \tag{16}$$

Here $\vec{r}\times\vec{p}$ equals $\vec{r}\times\mu\vec{v}$, since the vector potential (10) has only a radial component, and so (16) coincides with (12), when written in terms of the gauge-invariant, but noncanonical, velocity operator. However, the spectrum of (16) should be reexamined.

The interaction term in (15) depends explicitly on φ and is not single valued at $\varphi=0,\ 2\pi$; therefore, wave functions should not be single valued. Since the gauge transformation (9) is implemented in the quantum theory by a unitary operator,

$$U_{\rm NS}=\exp(-ie/\hbar c)\Omega_{\rm NS},$$

$$H_{\rm NS}=U_{\rm NS}HU_{\rm NS}^{-1}=\frac{1}{2\mu}\left(\vec{p}+\frac{e}{c}\nabla\Omega_{\rm NS}-\frac{e}{c}\vec{A}\right)^2=\frac{1}{2\mu}\left(\vec{p}-\frac{e}{c}\vec{A}_{\rm NS}\right)^2, \tag{17}$$

$$M_{\rm NS}=U_{\rm NS}MU_{\rm NS}^{-1}=\vec{r}\times\vec{p}+(e/2\pi c)\Phi,$$

eigenfunctions in the new gauge are related to previous ones by the multiplicative phase $\exp[(-ie/\hbar c)(\varphi/2\pi)\Phi]$, but the eigenvalues of angular momentum remain the same integers.

It should be clear that the description in the singular, short-range gauge (6), (7), when properly regulated as in (8), will also result in integer angular momentum eigenvalues, and wave functions will acquire a multiple-valued phase $\exp[(-ie/\hbar c)(\varphi/2\pi)\Phi_\infty]$, in the limit that the regularization is removed ($\alpha\to 0$).

Let us observe that in short-range gauges, singular or not, wave functions remain multiple valued even outside the solenoid $r>R$, because of the factor $\exp[(-ie/\hbar c)(\varphi/2\pi)\Phi_\infty]$. This may appear paradoxical, but we assert that the phase factor is not unnatural: The potential, present for $r>R$, is multiple valued, thus inducing multiple valuedness in the wave function, even in the region of vanishing $\vec{A}$, in which, however, there is nonvanishing flux.

If one asserts that in the long-range gauge, wave functions *are not* single valued, but acquire, after rotation by 2π, a phase $\exp[(ie/\hbar c)\Phi_\infty]$, then in short-range gauges wave functions *are* single valued outside the solenoid, and angular momentum eigenvalues become integers augmented by $(e/2\pi c)\Phi_\infty$. This is the anomalous angular momentum announced in the literature.[3]

However, the above is incorrect; not only is there no basis for abandoning single valuedness of wave functions in the presence of manifestly rotationally symmetric and regular, albeit long-range, potentials, but also the above violates quantum mechanical ideas and the correspondence principle, which asserts that quantized operators arise from classical adiabatic invariants. If angular momentum eigenvalues have a contribution proportional to Φ_∞, and the flux varies in time, they would be time dependent; this cannot be so. Also, a quantity with arbitrary time dependence cannot be an adiabatic invariant.

Moreover, as we demonstrate below, the vortex gives rise to half-integer angular momentum eigenvalues[3] only if wave functions are single valued in the regular, single-valued gauge.

Finally, let us consider the combined particle-field angular momentum in the regular, long-range gauge. In the absence of long-range effects this constant of motion takes a familiar form, which is obtained from the Belinfante, symmetric energy-momentum tensor:

$$M_B=\vec{r}\times\mu\vec{v}+c^{-1}\int d^2x\,\vec{x}\times(\vec{E}\times\vec{B})=\vec{r}\times\mu\vec{v}-c^{-1}\int d^2x\,\vec{x}\cdot\vec{E}B. \tag{18a}$$

The above differs from the canonical, Noether angular momentum M by a surface term:

$$M = M_B + c^{-1}\int d^2x\, \partial_i (E^i \epsilon^{nm} x^n A^m). \tag{18b}$$

However, the long-range nature of the gauge potential prevents us from using (18a) and (18b) interchangeably. Indeed one can show that M_B is not conserved when the flux is time dependent: $\dot{M}_B = (-e/2\pi c)\dot{\Phi}_\infty$. Also the surface term in (18b) does not vanish, and its contribution is precisely $(e/2\pi c)\Phi_\infty$, so that $M = M_B + (e/2\pi c)\Phi_\infty$ is constant. Substituting the solenoid magnetic field into (18) and using $\nabla\cdot\vec{E} = e\delta(\vec{r}-\vec{x})$ gives

$$M = \vec{r}\times\mu\vec{v} + (e/2\pi c)\Phi = \vec{r}\times\vec{p}, \tag{19}$$

which coincides with the particle-mechanics result (12).

Charged particle in the vortex field.—By the vortex, we shall mean static configurations of electromagnetic and complex scalar Higgs fields, where the electromagnetic field has the solenoid profile, with quantized total flux $\Phi_\infty = (\pi\hbar c/e)N$. The Higgs field χ, representing Cooper pairs, carries charge $2e$. In the regular long-range gauge, the profile of the Higgs field,

$$\chi(\vec{r}) = e^{iN\varphi} f(r);\quad N = 0, \pm 1, \dots, \tag{20}$$

is not rotationally invariant, since the phase carries angular dependence, but it is single valued and continuous. The function $f(r)$ vanishes at the origin, and rapidly approaches its asymptote f_∞ for $r > R$. A gauge transformation with gauge function Ω changes the Higgs field to

$$\chi \to \{\exp[-i(2e/\hbar c)]\Omega\}\chi \tag{21}$$

so that in the singular gauge (6), (7) the Higgs field is radially symmetric, $\chi_S = f(r)$, as is the gauge field.

The charged particle is described by a complex field Ψ, which is the only dynamical variable of the problem, governed by the following Lagrange density:

$$\mathcal{L} = i\hbar\Psi^* \frac{\partial}{\partial t}\Psi - \frac{\hbar^2}{2\mu}\left|\left(\nabla - \frac{ie}{\hbar c}\vec{A}\right)\Psi\right|^2 - \tfrac{1}{2}g\Psi^*\Psi^*\chi - \tfrac{1}{2}g\Psi\Psi\chi^*. \tag{22}$$

In addition to electromagnetic interactions, the particle experiences interaction with the scalar Higgs field, through the last two terms in (22), whose form is dictated by gauge invariance.

We substitute the vortex configuration (3) and (20) into (22) and inquire how the resulting Lagrangian, containing external gauge and Higgs fields, responds to rotations.

In a conventional Lagrangian, one expects that infinitesimal rotations are realized on the field by $\delta\Psi = -\omega\epsilon^{ij} r^i \partial_j \Psi = -\omega\,\partial\Psi/\partial\varphi$, and that in a rotationally invariant theory, the change in the Lagrange density is a total derivative, so that the Lagrangian is invariant: $\delta\mathcal{L} = -\omega\partial_i(\epsilon^{ij} r^j \mathcal{L})$, $\delta L = \int d^2r\,\delta\mathcal{L} = 0$.

However, with the background fields (3) and (20), the Lagrangian is not invariant against the conventional rotation transformation, because of the angular dependence of the Higgs background field. But if we supplement the usual rotation by a gauge transformation, as suggested in Ref. 3,

$$\bar{\delta}\Psi = -\omega\epsilon^{ij} r^i \partial_j \Psi + i\omega N\Psi/2 = -\omega(\partial/\partial\varphi - \tfrac{1}{2}iN)\Psi \tag{23}$$

then invariance holds: $\bar{\delta}\mathcal{L} = 0$. The consequent constant of motion follows by Noether's theorem, and is identified with the angular momentum:

$$M = \int d^2r\, \frac{\delta\mathcal{L}}{\delta\dot{\Psi}}\,\frac{\bar{\delta}\Psi}{\omega} = \int d^2r\,\Psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial\varphi} - \frac{\hbar N}{2}\right)\Psi. \tag{24}$$

(We require $\Psi^*\Psi$ to have the same value at $\varphi = 0$ and at $\varphi = 2\pi$, so that M is real.) Time dependence is governed by an equation which follows from (22),

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2\mu}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2\Psi + g\chi\Psi^*, \tag{25}$$

and M is conserved when Ψ^2 at $\varphi = 0$ is the same as at $\varphi = 2\pi$.

Note that the rotation generator (24) possesses, in addition to the kinematical term, a further contribution, which generates the gauge transformation that is necessary to compensate for the rotational noninvariance of the Higgs background field. But before concluding that the angular momentum is modified, we must evaluate fully each contribution to (24).

Time may be separated in (25) by a two-phase *Ansatz*:

$$\Psi(t, \vec{r}) = e^{-iEt/\hbar}\psi_1(\vec{r}) + e^{iEt/\hbar}\psi_2^*(\vec{r}). \tag{26a}$$

The resulting static equations,

$$E\psi_1 = (2\mu)^{-1}[\vec{p} - (e/c)\vec{A}]^2\psi_1 + g\chi\psi_2,$$
$$-E\psi_2 = (2\mu)^{-1}[\vec{p} + (e/c)\vec{A}]^2\psi_2 + g\chi^*\psi_1, \tag{26b}$$

are well known, and similar ones have been analyzed in the literature.[7] The eigenvalue problem is of the form $H\psi = E\sigma^3\psi$,

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

and the norm $\int d^2r\,\psi^\dagger\sigma^3\psi$, which is nonvanishing for sufficiently small g, can be set to ± 1. Eigenvalues are real, since H is Hermitian:

$$E = \int d^2r\,\psi^\dagger H\psi / \int d^2r\,\psi^\dagger\sigma^3\psi.$$

The angular dependence is separated by

$$\psi_1(\vec{r}) = (2\pi)^{-1/2}e^{i(n+N/2)\varphi}u_1(r),\quad \psi_2(\vec{r}) = (2\pi)^{-1/2}e^{i(n-N/2)\varphi}u_2(r); \tag{27a}$$

$$Eu_1 = \left(\frac{-\hbar^2}{2\mu}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{[\hbar(n+N/2)-(e/2\pi c)\Phi]^2}{2\mu r^2}\right)u_1 + gfu_2,$$
$$-Eu_2 = \left(\frac{-\hbar^2}{2\mu}\frac{1}{r}\frac{d}{dr}r\frac{d}{dr} + \frac{[\hbar(n-N/2)+(e/2\pi c)\Phi]^2}{2\mu r^2}\right)u_2 + gfu_1. \tag{27b}$$

Consistent with our treatment of the solenoid, wave functions should be single valued and continuous in the nonsingular, single-valued gauge that we are using, and indeed this is the choice made in the literature.[7] Consequently $m \pm \frac{1}{2}N$ must be an integer, which means that for odd N, m is half-integer. Finally we insert (27a) in (24) and find

$$M = \hbar m \int d^2r(|u_1|^2 - |u_2|^2) = \pm\hbar m. \tag{28}$$

The contribution from the gauge transformation has disappeared—as it must, since there *does* exist a rotationally symmetric, albeit singular, gauge—but the effect of a vortex with odd-integer flux remains in the half-integral quantum numbers for the angular momentum, confirming published results.[3] Note that this conclusion requires (1) the presence of the charged-particle–Higgs-field interaction, and (2) single valuedness and continuity of the wave functions. Indeed, if the vortex were treated with multiple-valued boundary conditions, only integer quantum numbers would occur.

We have also performed a completely dynamical, field-theoretical analysis of this problem, where in the fully relativistic theory we exploit the SO(2, 1) algebra to define unambiguously the angular momentum as the commutator of the Lorentz boost generators. This treatment of the vortex, using collective coordinates and expanding in powers of the electromagnetic coupling constant e, also shows that the angular momentum of a charged particle in the presence of a vortex with odd-integer flux is half-integer, provided that the charged-particle field is required to be single valued and continuous in the nonsingular long-range gauge.[8]

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[5]Here we depart from the three-dimensional situation, since in our two-dimensional description there is no possibility of including, in a current-free region, the return flux which is used in Ref. 4 to refute the conclusions of Ref. 3.

[6]If one considers a problem with an excluded region of space—a disk cut out of the origin—then there is no single-valuedness requirement on the wave function, even though the potential is single valued. Rather a Bloch condition is appropriate: $\Psi|_{\varphi+2\pi} = e^{i\lambda}\Psi|_\varphi$; so Ψ is given by $e^{i\lambda\varphi/2\pi}$ times a periodic function, and angular momentum is not quantized at all, since it can take arbitrary values: $\lambda/2\pi$ plus integer, as is appropriate to an operator conjugate to an unlimited variable $(-\infty \leq \varphi \leq \infty)$. This problem, as well as the solenoid which we discuss here, has been analyzed in the literature. See M. Peshkin, I. Talmi, and L. Tassie, Ann. Phys. (N.Y.) 12, 426 (1961); L. Tassie and M. Peshkin, Ann. Phys. (N.Y.) 16, 177 (1961); M. Peshkin, Phys. Rep. 80, 375 (1981); G. Goldin, R. Menikoff, and D. Sharp, J. Math. Phys. 22, 1664 (1981); Wilczek, Ref. 3.

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