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Quantum Statistics for Distinguishable Particles

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Quantum statistics can be reconciled with such classical ideas as distinguishable particles. Bose-Einstein and Fermi-Dirac statistics are derived for distinguishable particles by making an assumption regarding probabilities which is different from the traditional assumption but equally reasonable.

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Since the inception of modern quantum mechanics, there has been interest in reconciling quantum theory of measurement with classical ideas such as distinguishability, locality, and determinism.[1] One approach is to assume that the quantum state vector corresponds not to the particle itself, but to some statistical ensemble of particles. This interpretation immediately suggests remedies for the most disturbing aspects of quantum measurement theory.

Unfortunately, many conceptual and practical barriers impede the development of a consistent statistical interpretation in which quantum mechanical equations are viewed as describing ensembles of essentially classical particles. Among these is the fact that such classical particles are presumably to be treated as distinguishable. Yet, distinguishable particles obey Maxwell-Boltzmann statistics, whereas real particles obey Bose-Einstein or Fermi-Dirac statistics.

Here we show that distinguishable particles can in fact obey Bose-Einstein or Fermi-Dirac statistics. This result is surprising and fascinating in its own right. Moreover, the crucial step in deriving quantum statistics for distinguishable particles is a simple change in the usual assumptions regarding probabilities. Thus, our work suggests that a reconciliation of quantum mechanics with the classical idea of distinguishability must begin with a reassessment of traditional probability assumptions. Pitowsky[2] reached similar conclusions regarding the reconciliation of quantum mechanics with the principle of locality.

In a traditional derivation of quantum statistics,[3] one assumes that an electron will occupy each available state with equal probability weighting. Thus, all distinct configurations are equally likely, aside from constraints due to conservation laws. For N particles distributed among M discrete states, the probability of a set of occupancies, $\{n_i\}$, $i=1,\ldots,M$, is simply proportional to the number of distinct configurations corresponding to $\{n_i\}$. If one counts distinct configurations assuming distinguishable particles, one obtains Maxwell-Boltzmann statistics, with the probability

$$P\{n_i\}=M^{-N}N!/n_1!\,n_2!\cdots n_M!. \qquad (1)$$

With indistinguishable particles each $\{n_i\}$ defines a unique configuration, and one obtains Bose-Einstein statistics,

$$P\{n_i\}=N!(M-1)!/(N+M-1)! \tag{2}$$

independent of the particular $\{n_i\}$. If one further imposes the Pauli exclusion principle, Fermi-Dirac statistics are obtained.

Two crucial assumptions are made in deriving (1) and (2). First, each distinct configuration is taken to have equal probability. This assumption is (apparently) dictated by symmetry and simplicity. To obtain quantum statistics, one further assumes particles to be indistinguishable. Though contrary to familiar classical thinking, this assumption is accepted because it yields physically correct results for observable quantities.

Here we show that the latter assumption can be replaced with the more intuitive one of distinguishable particles, if we modify the assumption that all distinct configurations have fixed equal probability weighting. We instead assign each discrete state i an *arbitrary* probability weighting $\{w_i\}$, subject only to the fundamental restrictions on probabilities, $0 \leq w_i \leq 1$ and $\sum_{i=1}^{M} w_i = 1$. The observed statistical distribution is taken to be the average over all possible $\{w_i\}$. Thus, the Maxwell-Boltzmann distribution (1) is replaced by

$$P\{n_i\} = N!\left[\prod_{i=1}^{M}(n_i!)^{-1}\int_0^1 dw_i w_i^{\,n_i}\right]\delta\left(1-\sum_{i=1}^{M} w_i\right). \tag{3}$$

The integral can be evaluated easily by incorporating the constraint on w_i via the limits of integration, taking $0 \leq w_2 \leq 1-w_1$, etc. Each successive integral then has the form

$$\int_0^a w^m(a-w)^n dw$$

and (3) can be solved recursively with the result $P\{n_i\}=N!(M-1)!/(N+M-1)!$. This is exactly the Bose-Einstein result (2).

For the case of Fermi-Dirac statistics, the Pauli exclusion principle requires that $n_i=0$ or 1. Then (3) still gives $P\{n_i\}$ independent of $\{n_i\}$ (for all allowed values) as does a traditional analysis assuming *either* distinguishable or indistinguishable particles.

The assumption made here of a uniform random distribution of probability weightings may at first seem arbitrary. However, in the absence of prior knowledge it is arguably more natural (i.e., entails a weaker assumption) to take the probability weights of the states as arbitrary and random than equal. Equation (3) is merely the statistical average over ensembles of systems with independent arbitrary weights $\{w_i\}$.

We mentioned the possible significance of this result for a statistical interpretation of quantum mechanics. The ensemble average (3) immediately suggests a stochastic hidden-variable theory. Hidden-variable theories represent a major class of attempts to resolve problems in the interpretation of quantum mechanics.[4,5] Since there is no reason why such a theory should be factorizable, it need not have the undesirable features of deterministic hidden-variable theories (such as satisfying the Bell inequalities).[4]

On the other hand, our result has at least one interesting implication even within the framework of traditional quantum theory. Indistinguishability is often taken to refer simply to the counting assumption underlying the usual derivation of (2), viz., that configurations differing only by exchange of particles are not counted as distinct. Since we have shown that either set of counting assumptions can lead to quantum statistics, the statistics which particles obey cannot be a criterion for their distinguishability.

Pitowsky[2] recently showed that spin-$\frac{1}{2}$ statistics may be reconciled with the principle of locality by the introduction of a nontraditional model of probabilities. That result, together with our own, forcefully suggests that quantum mechanics may after all be compatible with classical ideas of locality and distinguishability; the key appears to lie in a reevaluation of underlying assumptions about probabilities.

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[1] See, for example, J. F. Klauser and A. Shimony, Rep. Prog. Phys. 41, 1881 (1978); J. Bub, *The Interpretation of Quantum Mechanics* (Reidel, Dordrecht, 1974).

[2] I. Pitowsky, Phys. Rev. Lett. 48, 1299 (1982).

[3] J. D. McGervey, *Introduction to Modern Physics* (Academic, New York, 1971).

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[5] B. S. DeWitt, Phys. Today 23, No. 9, 30 (1970).