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[19]The curious way in which the effect disappears is best analyzed by putting the system in a sphere of radius R. The integrand in (12) then contains a factor of $1-\exp[-2Rx]$. For any finite R, setting M equal to zero leads to zero charge. However, the Compton radius of the fermion is then larger than the system. If R is set equal to infinity before M is set equal to zero, the charge remains $-(e\theta/2\pi)$.

[20]By making use of the anomalous commutation relations (9a) or (9b), one easily sees how the phase $e^{-\theta}$ that results from a gauge transformation resides in the states as in Ref. 6. I also mention a simple derivation of Eq. (13). Namely, the expectation value of $\exp(-i \times Q_5\theta/2)Q\exp(+iQ_5\theta/2)$ in the monopole state for $\theta=0$ is $-e\theta/2\pi$ by (9). However, this is the expectation value to consider because of the boundary condition (4). Q_5 is not well defined for massless fermions.

[21]R. J. Eden, *High Energy Collision of Elementary Particles* (Cambridge Univ. Press, Cambridge, England, 1967).

[22]L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), p. 354. The additional contribution to the phase shift may also be interpreted as coming from the ambiguity in the asymptotic behavior of the D function in the N/D expression for the scattering amplitude. See S. C. Frautschi, *Regge Pole and S-Matrix Theory* (Benjamin, New York, 1963).

[23]B. Grossman, "Time Delay and the Dyon Charge" (to be published). In this paper we show how Levinson's theorem is equivalent to the Friedel sum rule which relates the dyon charge $e\theta/2\pi$ to time delay.

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Solution of the Infrared Problem

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(Received 13 August 1982)

The complete electromagnetic correction to a coordinate-space Feynman function is separated into a product of two factors. The first is a unitary operator that contains all contributions corresponding to the classical electromagnetic radiation field. The second is free of infrared divergences: It can be transformed into momentum space, and enjoys there the normal analytic properties. This result solves the infrared problem and maintains the physically correct asymptotic properties in coordinate space.

PACS numbers: 12.20.Ds, 11.10.Jj, 11.20.Fm

The well-known "infrared catastrophe" in quantum field theory consists of the following fact: The electromagnetic corrections to the S matrix are represented by integrals whose contributions from very soft photons often diverge. A way around this difficulty was indicated by Bloch and Nordsieck,[1] who showed, in some simple cases, that these infrared-divergent contributions cancel out of the expressions for the observable probabilities, provided the nonobservability of very soft photons is taken into account. The Bloch-Nordsieck observation has been generalized in a series of works that have culminated in the central work in this field, the paper of Yennie, Frautschi, and Suura (YFS).[2] The YFS paper gave lengthy arguments to support their contention that all of the infrared-divergent contributions to the S matrix can be collected into exponential factors that cancel out of the expressions for observable probabilities. However, at the end of a technical appendix to their paper YFS listed some of the difficulties with their arguments, and concluded that a rigorous proof of their conjecture would probably be prohibitively complicated. The difficulties with the YFS arguments are particularly serious when the S matrix is evaluated at a singularity.

The YFS infrared separation was used by Chung[3] to define an infrared-finite S matrix: infrared finiteness was (presumably) achieved by incorporating the YFS infrared factor into coherent initial or final states. This infrared-finite S matrix was examined by Storrow,[4] Kibble,[5] and Zwanziger,[6] who found that the pole singularity normally associated with a stable particle was converted by the effects of soft photons to a nonpole form.

Such a change in the character of the S-matrix singularities could be as catastrophic as the infrared divergence itself. For the physical interpretation of the theory depends on the fact that stable particles propagate over macroscopic distances in an essentially classical way, and on the fact that any process involving an intermediate stable particle behaves essentially like a combination of two separate processes. These properties depend normally[7,8] on the pole character of the one-particle exchange singularity, and on the factorization of the residue of that pole. Any change of these analyticity properties would jeopardize the ability of the theory to accomodate stable (or nearly stable) charged particles.

These difficulties can be overcome by a new approach to the infrared-divergence problem that separates out the classical electromagnetic effects already in coordinate space. For terminological convenience, and greater generality, we consider not QED itself but rather the electromagnetic corrections to a given (perhaps nonelectromagnetic) process represented by a Feynman diagram D. Let $F^D(x)$ be the coordinate-space Feynman function corresponding to D. Here $x=(x_1, \ldots, x_n)$ specifies the space-time locations of the n vertices of the space-time diagram $D(x)$ that is topologically equivalent to D.

Suppose, first, that the diagram D has no charged external lines. Then $F^D(x)$ is transformed by the inclusion of all electromagnetic interactions with D to the operator-function

$$\hat{F}_{\mathrm{op}}{}^D(x) = U(L(x))\tilde{F}_{\mathrm{opr}}{}^D(x). \tag{1}$$

Here $U(L(x))$ is a unitary operator in photon space that depends on the set $L(x)$ of space-time polygons formed by the charged lines of $D(x)$. The explicit form of $U(L(x))$ is

$$U(L(x)) = \exp[\langle a^* \cdot J(L(x))\rangle] \exp[-\langle a \cdot J^*(L(x))\rangle] \exp[i\Phi(L(x)) - \tfrac{1}{2}\langle J^*(L(x)) \cdot J(L(x))\rangle], \tag{2}$$

where $a^*(k)$ and $a(k)$ are photon creation and annihilation operators,

$$\langle f \cdot g\rangle = \int \frac{d^4k}{(2\pi)^4}\, 2\pi\delta^+(k) f_\mu(k)(-g^{\mu\nu}) g_\nu(k), \tag{3}$$

$$J_\mu(L(x),k) = -ie\int_{L(x)} dx_\mu{}' e^{ikx'}, \tag{4}$$

and

$$\Phi(L(x)) = \frac{(-ie)^2}{8\pi}\int_{L(x)}\int_{L(x)} dx'\,dx''\,\delta((x'-x'')^2). \tag{5}$$

The (Coulomb) phase $\Phi(L(x))$ is the classical action associated with the motion of a charged particle around the closed loops $L(x)$. The remaining part of $U(L(x))$ acting on the photon vacuum creates the coherent photon state that corresponds to the classical electromagnetic field radiated by the motion of a charged particle around the closed loops $L(x)$.

The quantity $\tilde{F}_{\mathrm{opr}}{}^D(x)$ is likewise an operator in photon space that depends on x. It gives no infrared difficulties: It can be Fourier transformed into momentum space $(x \to p)$, and the dominant singularities of all (multi-)photon momentum-space matrix elements $\langle k'|\tilde{F}_{\mathrm{opr}}{}^D(p)|k''\rangle$ of $\tilde{F}_{\mathrm{opr}}{}^D(p)$ have the same character as the corresponding singularities in a theory with no massless particles. Moreover, the discontinuities around these singularities are infrared finite.

Formula (1) is derived by separating the photon coupling into its "classical" and "quantum" parts: The coupling of a photon of momentum-energy k into a charged line i of D is expressed in the form

$$-ie\gamma_\mu = C_\mu(k,z_i) + Q_\mu(k,z_i), \tag{6}$$

where z_i is the space-time difference between the two end points of the line segment i of $D(x)$, and

$$C_\mu(k,z) = -iez_\mu \not{k}(k\cdot z)^{-1}. \tag{7}$$

The operator $U(L(x))$ is the full contribution from all classical photons, which are those that couple into $L(x)$ only via the classical couplings. The operator $\tilde{F}_{\mathrm{opr}}{}^D(x)$ is the full contribution from all quantum photons, which are those that couple into $L(x)$ with a quantum coupling on at least one end. The function $\tilde{F}_{\mathrm{opr}}{}^D(x)$ is constructed as the sum of Feynman-like contributions from all the different ways that photon lines can be connected to D. The rules for computing $\tilde{F}_{\mathrm{opr}}{}^D(x)$ are the same as the Feynman rules, except that one must use the coupling $Q_\mu(k,z_i)$—and sometimes $C_\mu(k,z_i)$—in place of $-ie\gamma_\mu$, and must use for the propagator of a photon with energy-momentum k emitted by a quantum coupling Q and absorbed by a classical coupling C the retarded propagator defined by replacing Feynman's denominator $k^2+i\epsilon$ by $(k^0+i\epsilon)^2-\vec{k}^2$. This modification of the propagator, together with the fact that the quantum coupling into a mass-shell line is linear in k, renders infrared finite the functions $\langle k' | \times \tilde{F}_{\mathrm{opr}}(p) | k'' \rangle$ and their discontinuities.

The physical observables are the transition probabilities associated with spatially localized free-particle wave functions $\psi_i(x_i)$ of the initial and final particles of D. Folding these wave functions into $\hat{F}_{\mathrm{op}}{}^D(x)$ gives the transition operator-amplitude $T_{\mathrm{op}}{}^D[\psi]$.

If there were no difficulties arising from massless particles then one could define the transition amplitudes for processes having charged-particle external lines by considering these processes to be subprocesses of a larger process D having no charged-particle external lines. The extraction could be effected in either one of two ways: (1) Take the residues of the pole singularities associated with those internal lines of D that are external lines of the subprocesses, or (2) shift the wave packets of the external particles of $T_{\mathrm{op}}{}^D[\psi]$ to infinity in a way such that the full process represented by D is decomposed into subprocesses that occur in separate space-time regions that are shifted to infinity in a way compatible with the idea that the intermediate charged particles travel like classical particles between these separated space-time regions. The latter procedure is the physical one, but its equivalence to the pole-residue method is assured by the normal connection mentioned earlier between momentum-energy singularities and space-time asymptotic behavior.[7,8]

These two ways of deriving the transition amplitudes for processes with charged-particle external lines can be used also in the presence of massless photons. In the space-time approach the external-particle wave functions $\psi \equiv \psi^\lambda$ are shifted to infinity as $\lambda \to \infty$ in a way such that the main contributions to the transition amplitude $T_{\mathrm{op}}{}^D[\psi^\lambda]$ come only from points $x = (x_1 \ldots, x_n)$ near the point $\lambda X = (\lambda X_1, \ldots, \lambda X_n)$. That is, the localization of the wave packets $\psi_i{}^\lambda$ renders negligible (in an appropriate mathematical sense) the contributions to the amplitude from points x not close to λX.

This effective confinement of x to a neighborhood of λX entails that the contributions to the observable probabilities of classical photons with $k \in \Omega$ become negligible as the size of Ω tends to zero. In particular, let Ω be a small neighborhood of $k = 0$ and let $U_\Omega(L(x))$ and $U^\Omega(L(x))$ represent the operators obtained from $U(L(x))$ by restricting the k integration to Ω, and to the complement of Ω, respectively. [The δ function in (5) must be Fourier analyzed to effect this separation.] Then

$$U(L(x)) = U_\Omega(L(x))U^\Omega(L(x)) \tag{8}$$

$$= U_\Omega(L(\lambda X))U^\Omega(L(x))U_\Omega(L(\lambda X))[U_\Omega{}^{-1}(L(\lambda X))U_\Omega(L(x)) - 1]U^\Omega(L(x)). \tag{9}$$

Suppose Ω is confined to the space corresponding to undetectable photons. Observable probabilities involve sums over all final states of the undetectable photons. Thus in the calculation of observable probabilities the unitary operator $U_\Omega(L(\lambda X))$ in (9) drops out. Hence in the contribution to observable probabilities coming from the first term in (9), and the corresponding contribution from $U^\dagger(L(x))$, the classical photons with $k \in \Omega$ have no effect at all. On the other hand, for sufficiently small Ω the contributions from the second term in (9) can be shown to be negligible due to the near equality of $U_\Omega(L(\lambda X))$ and $U_\Omega(L(x))$. The very soft classical photons have, therefore, essentially no effect on probabilities and the infrared contributions arise only from $\tilde{F}_{\mathrm{opr}}{}^D(p)$. The normal analytic structure and infrared finiteness of the functions $\langle k' | \tilde{F}_{\mathrm{opr}}{}^D(p) \times | k'' \rangle$ and their discontinuities thus ensure that the observable probabilities enjoy the asymptotic behavior that corresponds to stable charged particles.

The same arguments show that if the initial $k \in \Omega$ photon states are Fock states then the main $k \in \Omega$ contribution to the asymptotic probabilities comes from final states that are $U_\Omega(L(\lambda X))$ times Fock states. Thus to define amplitudes that contribute significantly to the asymptotic behavior of probabilities one should use these coherent final states. But then the operator $U_\Omega(L(\lambda X))$ in (9) is cancelled, and the earlier arguments now show that the very soft classical photons have negligible contribution to the amplitudes. Then the normal analytic structure of the functions $\langle k' | F_{\mathrm{opr}}(p) | k'' \rangle$ ensures that infrared-finite amplitudes for processes with charged-particle external lines can be equivalently defined by the factorization of residues of charged-particle poles or by the factorization of asymptotic amplitudes. Details are given elsewhere.[9,10]

The work described here was begun in collaboration with Peter Holland, and was completed with the participation of T. Kawai. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy and Nuclear Physics, of the U. S. Department of Energy under Contract No. DE-AC03-76SF00098.

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