Does a Dyon Leak?

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In the presence of a *CP* nonconservation measured by an angle θ, the ground state of a point magnetic monopole is shown to have an electric charge of value $-e\theta/2\pi$ which changes discontinuously to zero for massless fermions. A new version of Levinson's theorm is also given. The latter effect as well as the *S*-wave helicity filp of a dyon can be interpreted as a leak at the origin.

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The theory of the magnetic monopole is now over fifty years old,[1,2] and despite the fact that conclusive proof of the existence of the monopole is still lacking,[3] recently there have been many interesting surprises that have arisen in grand unified theories that include monopoles in their spectrum.[4-9] These new discoveries have led us to reconsider the scattering of fermions by a point Abelian magnetic monopole. I discuss in this paper the physics of the lowest partial-wave scattering in the presence of a *CP*-nonconserving angle. I first review how the helicity flip of *S*-wave scattering[10] can be interpreted as a "leak" at the origin. Then I find that the monopole ground state has an electric charge[8] $-(e\theta/2\pi)$ which changes discontinuously to zero when the fermion mass is zero. I also find a new version of Levinson's theorem[11] for the phase shifts. Because the equation for the $J=0$ radial fermionic wave function is essentially the same whether the monopole is Abelian or non-Abelian, the present results apply to both cases except for the discussion of *S*-wave helicity flip.[10] In the non-Abelian case, the charge or the helicity can change sign.[9,12]

As explained several years ago,[10] the Hamiltonian for the interaction of a point Abelian monopole with a massive fermion is not self-adjoint. However, the theory of deficiency indices[13] can be used to show that there is a one-parameter family of self-adjoint extensions determined by the value of the wave function at the origin. If one adds to the Hamiltonian for the above system a small anomalous magnetic moment, $(Ke/2M)\vec{\sigma}\cdot\vec{B}$, the new Hamiltonian is self-adjoint.[10,14] The new Hamiltonian has zero deficiency indices and the wave functions for the fermion vanish at the origin like $\exp[-K|q|(2Mr)^{-1}]$. We will consider a more general Hamiltonian with *CP* nonconservation so that the boundary condition on the wave function at the origin depends on the angle θ measuring the *CP* nonconservation.[15] This boundary condition then chooses one self-adjoint extension of the original Hamiltonian with $K=0$ and no explicit *CP* nonconservation.

Following Wu and Yang,[16] we can write the Hamiltonian for the lowest partial wave, with $J=|q|-\frac{1}{2}$, of a fermion with mass M interacting with a point magnetic monopole as

$$H(K,\theta)=-i\gamma_5 d/dr-\beta\exp(i\theta\gamma_5)M+K|q|\beta\exp(i\theta\gamma_5)(2Mr^2)^{-1},\quad \gamma_5=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\tag{1}$$

operating on a wave function

$$r\psi=\begin{pmatrix}F(r)n_m\\G(r)n_m\end{pmatrix},\tag{2}$$

where n_m is the relevant monopole harmonic[14,16] specified by the values of

$$\vec{J}=\vec{r}\times(-i\nabla-e\vec{A})-q\vec{r}/r+\tfrac{1}{2}\vec{\sigma}\tag{3}$$

and $q=eg$ which can be chosen to be positive without loss of generality. We find that

$$G(0)/F(0)=i\tan(\tfrac{1}{2}\theta-\tfrac{1}{4}\pi)\tag{4}$$

is the boundary condition for this model which we want to impose on the wave functions for the Hamiltonian $H_0=H(0,\pi)$.

The eigenfuntions of H_0 with eigenvalues E consist of scattering states, ψ_E, and a bound state, ψ_B, for $\cos\theta < 0$:

$$\psi_E = \frac{N}{r}\begin{pmatrix} [S(k)e^{ikr} + e^{-ikr}]n_m \\ [k/(E+M)][S(k)e^{ikr} - e^{-ikr}]n_m \end{pmatrix}, \tag{5}$$

$$N = \frac{1}{2}\left(\frac{k}{E - M\sin\theta}\right)^{1/2}\left[\cos\left(\frac{\theta}{2} - \frac{\pi}{4}\right) - i\sin\left(\frac{\theta}{2} - \frac{\pi}{4}\right)\frac{E+M}{k}\right], \quad E^2 = k^2 + M^2 \tag{6}$$

with the S-matrix[17] $S(k)$ defined in terms of the Jost function $f(k)$ as follows:

$$S(k) = f(k)/f(-k),$$
$$f(k) = \left(\frac{2(E - M\sin\theta)}{(E-M)(1-\sin\theta)}\right)^{1/2} e^{i\delta(k)}, \tag{7}$$
$$\delta(k) = -\tan^{-1}\left[\frac{E+M}{k}\frac{(1-\sin\theta)}{\cos\theta}\right].$$

ψ_E is normalized with measure $\pi^{-1}dE$. For $\cos\theta < 0$, there is a bound state, corresponding to $f(k) = 0$:

$$\psi_B = \frac{(2B)^{1/2}}{r}\exp\left[i\left(\frac{\theta}{2} - \frac{\pi}{4}\right)\gamma_5\right]\begin{pmatrix} n_m \\ 0 \end{pmatrix}\exp(-B_r), \tag{8}$$
$$E = M\sin\theta, \quad B = -M\cos\theta.$$

The S matrix is manifestly unitary and dependent on the angle θ. For S-wave scattering, we have $J=0$, and as Goldhaber[10] observed, the scattering is pure helicity flip because $q = \frac{1}{2}\vec{\sigma}\cdot\hat{r}$. Nevertheless, the scattering amplitude is nonzero and independent of angle. This phenomenon occurs because the helicity operator, which can be written as $-i\,d/dr$, has no self-adjoint extensions.[10] Defined on a finite interval $[0,R]$, the operator $-i\,d/dr$ can be interpreted as a self-adjoint translation operator which conserves probability on functions satisfying $\varphi(R) = e^{i\alpha}\varphi(0)$. However, because the helicity operator must be defined on an infinite interval $[0,\infty)$, helicity can leak at the origin. In the case of the non-Abelian monopole, the charge of the fermion can change sign.[9,12] This suggests that the U(1) charge of the monopole may not be well defined. In fact, the anomalous commutation relation[18] between the charge density and the axial charge density,

$$[J_0(x), J_0^5(y)] = (-ie/2\pi^2)\vec{B}\cdot\vec{\partial}_x\,\delta^3(\vec{x} - \vec{y}) \tag{9a}$$

or

$$[Q, Q^5] = 2iq/\pi, \tag{9b}$$

demonstrates that the non-Abelian monopole interacting with fermions is not in a well-defined charge state when it is in a well-defined helicity state and vice versa.

Returning to the Abelian monopole, one can calculate the electric charge of the ground state, which we take to include the bound state when it exists:

$$Q = (e/\pi)\int d^3x\left[\int_{-\infty}^{-M} dE\,\psi_E^\dagger\psi_E + \psi_B^\dagger\psi_B\right] \tag{10}$$

$$= \frac{e}{\pi}\int dr\left\{\int_{-\infty}^{-M} dE\left[\frac{E}{k} + \frac{M}{2k}e^{2ikr}\left(\frac{-M + E\sin\theta - ik\cos\theta}{E - M\sin\theta}\right) + \text{c.c.}\right] + 2Be^{-2Br}\right\}. \tag{11}$$

The first term is thrown away because it is just the contribution of the vacuum. Then turn the integral into an integral over dk that we close in the upper half plane. Using Cauchy's theorem one finds two contributions to the integral. First, there is a pole at $k = -iM\cos\theta$, whose residue exactly cancels the bound-state term for $\cos\theta < 0$ (otherwise there is no pole and no bound state). There is also a contribution from a cut running from iM to $i\infty$ which can be expressed as follows:

$$Q = -\frac{eM\sin\theta}{\pi}\int_M^\infty \frac{dx}{(x^2 - M^2)^{1/2}(x + M\cos\theta)} \tag{12}$$

$$= -e\theta/2\pi. \tag{13}$$

The monopole is really a dyon, as Witten[8] first observed. However, an examination of (11) or (12) shows that the effect disappears discontinuously for $M = 0$.[19] At this point, the S matrix is just a nondynamical phase, $-ie^{i\theta}$. However, the wave functions depend on θ in an explicit breaking of chiral invariance[20] by the boundary condition (4)

$$\psi_E = \exp(\tfrac{1}{2}i\theta\gamma_5)\begin{pmatrix} \cos(kr - \pi/4) \\ i\sin(kr - \pi/4) \end{pmatrix}. \tag{14}$$

The presence of the bound state for this system suggested an examination of Levinson's theorem[11] that relates the phase shift to the number of bound states. In the application of this theorem to field theory,[21] one must take into account not only bound-state poles, but also cuts like those found in the integrand (12). We find the following relation:

$$\int_{\infty}^{-M} \delta'(E)dE + \int_{M}^{\infty} \delta'(E)dE = -\theta - \pi, \qquad (15)$$

$$\delta'(E) = \delta'(k)\,dk/dE \qquad (16a)$$

$$= +M\cos\theta/2k(E - M\sin\theta). \qquad (16b)$$

Equation (16b) clearly shows the presence of the bound-state pole, which contributes the term π, as well as the cuts which contribute a total of $-\theta$. The dyon causes an additional change of the phase shift by $-\theta$ just as a zero-energy, $l=0$ bound state causes an additional change of the phase shift by $\frac{1}{2}\pi$ because the wave function can "leak" out.[22] Once again, the effect also disappears discontinuously when $M=0$.

We can give an intuitive argument for the change in Levinson's theorem when the theory is not time-reversal invariant. Since the derivative of the phase shift with respect to energy can be interpreted as a time delay,[22,23] the left-hand side of (15) can be interpreted as the difference in time delays resulting from the scattering off all the positive-energy states minus the time delays from scattering off all the negative-energy states. The result is proportional to the amount of time-reversal noninvariance plus the number of bound states (because they occur asymmetrically in the complex plane). At $\theta = -\pi$, there is complete symmetry between the particles and the Dirac sea.

Finally, we mention that we could have modified the Hamiltonian H_0 to obtain a self-adjoint operator by extending the magnetic charge of the monopole over a region of size R_0 as well as by adding an anomalous magnetic moment K. For the Abelian monopole, one must define the order in which K and R_0 approach zero. In light of the recent suggestion that a small anomalous magnetic moment may suppress baryon catalysis,[24] it is important to examine the order of the limits for the extended non-Abelian monopole. I hope to return to this issue.

I would like to thank my colleagues at Rockefeller University for illuminating and encouraging discussions. In particular, I would like to thank Professor M. A. B. Bég for urging me to publish my results and Professor N. N. Khuri for suggesting that I examine Levinson's theorem. I also thank Professor C. Callias for a discussion. This work was supported in part by the U. S. Department of Energy under Grant No. DE-AC02-82ER40033.B000.

After this paper was submitted, H. Yamagishi brought to my attention his report.[25] I thank him for bringing to my attention his interesting work.

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[17]The scattering amplitude has two cuts in the complex k^2 plane: one on the right-hand side of the real axis starting at $k^2=0$ and one on the left-hand side starting at $k^2=-M^2$. When there is no bound state, the latter cut indicates, in the effective-range approximation, that the range of the interaction is M^{-1}. It is the long range of the interaction for small M that Ref. 7 makes use of to catalyze baryon decay at large rates. When there is a bound state that is close to the right-hand cut, resonance scattering will dominate. We thank Professor N. N. Khuri for this observation.

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[19]The curious way in which the effect disappears is best analyzed by putting the system in a sphere of radius R. The integrand in (12) then contains a factor of $1-\exp[-2Rx]$. For any finite R, setting M equal to zero leads to zero charge. However, the Compton radius of the fermion is then larger than the system. If R is set equal to infinity before M is set equal to zero, the charge remains $-(e\theta/2\pi)$.

[20]By making use of the anomalous commutation relations (9a) or (9b), one easily sees how the phase $e^{-\theta}$ that results from a gauge transformation resides in the states as in Ref. 6. I also mention a simple derivation of Eq. (13). Namely, the expectation value of $\exp(-i \times Q_5\theta/2) Q \exp(+iQ_5\theta/2)$ in the monopole state for $\theta=0$ is $-e\theta/2\pi$ by (9). However, this is the expectation value to consider because of the boundary condition (4). Q_5 is not well defined for massless fermions.

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Solution of the Infrared Problem

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The complete electromagnetic correction to a coordinate-space Feynman function is separated into a product of two factors. The first is a unitary operator that contains all contributions corresponding to the classical electromagnetic radiation field. The second is free of infrared divergences: It can be transformed into momentum space, and enjoys there the normal analytic properties. This result solves the infrared problem and maintains the physically correct asymptotic properties in coordinate space.

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The well-known "infrared catastrophe" in quantum field theory consists of the following fact: The electromagnetic corrections to the S matrix are represented by integrals whose contributions from very soft photons often diverge. A way around this difficulty was indicated by Bloch and Nordsieck,[1] who showed, in some simple cases, that these infrared-divergent contributions cancel out of the expressions for the observable probabilities, provided the nonobservability of very soft photons is taken into account. The Bloch-Nordsieck observation has been generalized in a series of works that have culminated in the central work in this field, the paper of Yennie, Frautschi, and Suura (YFS).[2] The YFS paper gave lengthy arguments to support their contention that all of the infrared-divergent contributions to the S matrix can be collected into exponential factors that cancel out of the expressions for observable probabilities. However, at the end of a technical appendix to their paper YFS listed some of the difficulties with their arguments, and concluded that a rigorous proof of their conjecture would probably be prohibitively complicated. The difficulties with the YFS arguments are particularly serious when the S matrix is evaluated at a singularity.

The YFS infrared separation was used by Chung[3] to define an infrared-finite S matrix: infrared finiteness was (presumably) achieved by incorporating the YFS infrared factor into coherent initial or final states. This infrared-finite S matrix was examined by Storrow,[4] Kibble,[5] and Zwanziger,[6] who found that the pole singularity normally asso-