Continuum Limit of Supersymmetric Field Theories on a Lattice

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The ground-state energy density E_0 is examined for a wide class of two-dimensional supersymmetric quantum field theories by use of a strong-coupling lattice expansion. It is shown that regardless of how the lattice expansion is defined (there are many ways to put the fermions on the lattice) $E_0 \to 0$ in the continuum limit.

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It is known[1] that the two-dimensional quantum field theory whose Euclidean Lagrangian in the presence of external sources is

$$L_E = \tfrac{1}{2}(\nabla_\mu \varphi)^2 + \tfrac{1}{2}\bar\psi\gamma_\mu\nabla_\mu\psi + \tfrac{1}{2}gS'(\varphi)\bar\psi\psi + \tfrac{1}{2}c^2g^2[S(\varphi)]^2 - J\varphi - \bar\eta\psi, \quad S'(\varphi) = dS/d\varphi, \tag{1}$$

is supersymmetric when $c=1$. Here, ψ is a two-component Majorana spinor and φ is a real spinless boson field; $S(\varphi)$ is an arbitrary function of φ.

The purpose of this paper is to examine the strong-coupling lattice expansion of such a theory. Clearly, since Lorentz symmetry is broken on a lattice, so is supersymmetry. However, we show that when $S(\varphi)$ is an odd function of φ, the supersymmetry is restored in the continuum limit. We demonstrate this by calculating the ground-state energy density E_0 as a sequence of extrapolants and showing that as $c \to 1$ each term in this sequence approaches zero. We interpret this result as a signal that supersymmetry is restored. The connection between the vanishing of the ground-state energy density and the preservation of supersymmetry is discussed in detail by Witten.[2]

We calculate the ground-state energy density E_0 from the vacuum persistence function Z written as a functional integral:

$$Z = \int D\varphi\, D\psi \exp\left(-\int d^2x\, L_E\right). \tag{2}$$

If Z is properly normalized,[3] then it can be written as

$$Z = \exp(-VE_0), \tag{3}$$

where V is the volume of the two-dimensional Euclidean space ($V = \int d^2x$).

Following the usual Lagrangian strong-coupling expansion approach[4] we separate off the kinematic terms in Z:

$$Z = \exp\left[\iint d^2x\, d^2y \left(\frac{1}{2}\frac{\delta}{\delta J(x)}\nabla^2\delta(x-y)\frac{\delta}{\delta J(y)} + \frac{\delta}{\delta\eta(x)}\gamma_\mu\nabla_\mu{}^x\delta(x-y)\frac{\delta}{\delta\bar\eta(y)}\right)\right]W, \tag{4}$$

where

$$W = \int D\varphi\, D\psi \exp\int d^2x \{-\tfrac{1}{2}gS'(\varphi)\bar\psi\psi - \tfrac{1}{2}c^2g^2[S(\varphi)]^2 + J\varphi + \bar\eta\psi\}. \tag{5}$$

We evaluate W by introducing a two-dimensional square lattice:

$$\int d^2x \to a^2\sum_i, \quad \int D\varphi \to \prod_{i=1}^{N}\int_{-\infty}^{\infty} d\varphi_i/(2\pi)^{1/2}, \quad \int D\psi \to \prod_{i=1}^{N} a^{-1}\int d\psi_i, \quad V = Na^2, \tag{6}$$

where a is the lattice spacing, N is the number of lattice points, and φ_i is the value of $\varphi(x)$ at the ith lattice point x_i. On the lattice W has the form

$$W=\prod_i a^{-1}(2\pi)^{-1/2}\int d\psi\int_{-\infty}^{\infty} d\varphi\, \exp\left(a^2\{-\tfrac{1}{2}gS'(\varphi)\bar\psi\psi-\tfrac{1}{2}c^2g^2[S(\varphi)]^2+J_i\varphi+\bar\eta_i\psi\}\right). \tag{7}$$

We evaluate the fermion integral in (7) using the usual Grassman rules and obtain

$$W=\prod_i ag(2\pi)^{-1/2}\int_{-\infty}^{\infty} d\varphi\, S'(\varphi)\exp\{\tfrac{1}{2}\bar\eta_i[1/a^2gS'(\varphi)]\eta_i-\tfrac{1}{2}c^2a^2g^2[S(\varphi)]^2+J_i\varphi a^2\}, \tag{8}$$

where we have used

$$\int d\psi\, \exp(-\tfrac{1}{2}\bar\psi A\psi)=[\mathrm{Det}(A)]^{1/2}=A.$$

For simplicity we specialize to the case

$$S=\varphi^{2k+1},\quad k\ \text{integer}, \tag{9}$$

although our conclusions hold when S is any odd function.[5] The integral in (8) can be simplified by using $\int_{-\infty}^{\infty}d\varphi=2\int_0^{\infty}d\varphi$. Note that only the even part of $\exp(J_i\varphi a^2)$ contributes to the integral and so we must replace $\exp(J_i\varphi a^2)$ by $\cosh(J_i\varphi a^2)$. Next, we make a change of integration variable:

$$agS(\varphi)=y. \tag{10}$$

The result is

$$W=\prod_i(2/\pi)^{1/2}\int_0^{\infty}dy\,\exp[-\tfrac{1}{2}c^2y^2+(4k+2)^{-1}\bar\eta_i\eta_i a^{-1}(ag)^{-1/(2k+1)}y^{-2k/(2k+1)}]\cosh[J_i(Y/ag)^{1/(2k+1)}a^2]. \tag{11}$$

The usual strong-coupling procedure is to expand the integrand in (11) as a series in powers of the dimensionless small parameter

$$x\equiv(ag)^{-1/(2k+1)}, \tag{12}$$

and then to integrate this series term by term in y. We obtain an expansion for W having the form

$$W=\prod_i\left(c^{-1}+\sum_{n=1}^{\infty}b_nx^n\right), \tag{13}$$

where the coefficients b_n depend on $\bar\eta_i\eta_i/a$ and a^2J_i.

It is important to point out that the first term in the series (13) for an ordinary field theory contains the parameters g and a. For example, for the anharmonic oscillator, the first term is[3] $2\pi^{1/2}a^{3/4}g^{1/4}\Gamma(\frac{1}{4})$. The special structure of this mixed fermion-boson theory on the other hand gives rise to the numerical constant $1/c$, independent of any dimensional parameters in the theory. It is crucial for the vanishing of the ground-state energy that the leading term in (13) approaches unity as the continuum theory becomes supersymmetric ($c\to1$).

The (connected) vertices for the lattice diagrams are obtained by expanding the logarithm of (13):

$$W=\exp[N\ln(c^{-1})+\textstyle\sum(\text{vertices})]. \tag{14}$$

Comparing (3) with (4) and (14) gives the graphical expansion for the ground-state energy density E_0:

$$-Na^2E_0=N\ln(c^{-1})+\textstyle\sum(\text{connected graphs}). \tag{15}$$

We are not concerned with the specific choice for the boson and fermion lines. Any definition gives a series for the connected graphs of the form $N\sum_{n=1}^{\infty}d_nx^n$. Thus, on the lattice E_0 has the form

$$E_0=a^{-2}\left(\ln c-\sum_{n=1}^{\infty}d_nx^n\right)=g^2x^{4k+2}\sum_{n=0}^{\infty}e_nx^n, \tag{16}$$

where $e_0=\ln c$ and $e_n=-d_n$, $n\geq1$.

It is necessary to extrapolate the series in (16) to the continuum. The continuum limit $a\to0$, $x\to\infty$, is performed by converting (16) into a sequence of extrapolants Q_N which converge to the continuum value of E_0.[4,6] The Nth extrapolant for any series of the form

$$x^{\alpha}\sum_{n=0}^{\infty}e_nx^n$$

is

$$Q_N=h_N^{-\alpha/N}, \tag{17}$$

where h_N is the coefficient of x^N in $(\sum_{n=0}^{\infty}e_nx^n)^{-N/\alpha}$.

It is easy to show that if $e_0\to0$ then Q_N in (17) approaches zero as $e_0^{1+\alpha}$. Thus, for all N, we conclude that the Nth extrapolant to E_0/g^2 in (16) vanishes like $(\ln c)^{4k+2}$ as $c\to1$ (the supersymmetric limit).[7]

In summary, we have easily obtained the result that $E_0=0$ by taking a supersymmetric limit ($c\to1$) of the nonsupersymmetric theory of (1). (Because of our choice of limits we obtain the result that supersymmetry is restored in the continuum without ever specifying a particular lattice version of the fermion inverse propagator.) Had we interchanged the limits $a \to 0$, $c \to 1$ and started with the $c = 1$ theory we would have found that the continuum extrapolants were all nonzero. To actually do such a calculation we would have to choose a particular method for treating lattice fermions. We believe, on the basis of our earlier work (Ref. 7), that the resulting sequence of extrapolants will converge to zero slowly. The best way, in that case, to show that the ground-state energy density vanishes is to show by direct calculation that the critical index for the lattice series is positive.

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[1]See, for example, L. Alvarez-Gaume, D. Z. Freedman, and M. T. Grisaru, Harvard University Report No. HUTMP 81/B111, 1981 (to be published).

[2]E. Witten, Nucl. Phys. B188, 513 (1981). Witten also shows that for a one-dimensional version of Eq. (1), that is, $n = 1$ supersymmetric quantum mechanics, when the superpotential S is even (odd) supersymmetry is broken (unbroken).

[3]A discussion of the normalization of functional integrals used in strong-coupling expansions is given in Sec. II of C. M. Bender, F. Cooper, G. S. Guralnik, D. H. Sharp, R. Roskies, and M. L. Silverstein, Phys. Rev. D 20, 1374 (1979).

[4]See, for example, C. M. Bender, F. Cooper, G. S. Guralnik, and D. H. Sharp, Phys. Rev. D 19, 1865 (1979). A discussion of Hamiltonian strong-coupling methods can be found in, e.g., S. Elitzur *et al.*, CERN Report No. TH 3389-CERN (unpublished); T. Banks and P. Windey, Nucl. Phys. B198, 226 (1982); V. Rittenberg and S. Yankielowicz, to be published; M. Peskin, unpublished, and references therein.

[5]When $S(\varphi)$ is an even function, the procedure used here and in Ref. 3 for generating a strong-coupling series fails because the logarithm defining connected vertices does not possess a Taylor expansion [see Eq. (14)]. This occurs because our strong-coupling expansion procedure is based on removing the fermion kinetic energy from the Lagrangian. The resulting path integral W in (7) in the absence of sources vanishes when $S(\varphi)$ is even. In one dimension it is possible to skirt this difficulty because one can perform the full fermion path integral in (2) which includes the fermion kinetic energy (see F. Cooper and B. Freedman, to be published). In higher dimensions this calculation has not yet been done but we believe that when it is done it will be shown that supersymmetry is broken when $S(\varphi)$ is even.

[6]C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, Phys. Rev. Lett. 43, 537 (1979).

[7]This result is consistent with that obtained for a one-dimensional version of Eq. (1), C. M. Bender, F. Cooper, and A. Das, "New Test for Spontaneous Breakdown of Supersymmetry" (to be published).