PHYSICAL REVIEW LETTERS

VOLUME 50 — 7 FEBRUARY 1983 — NUMBER 6

Propagating Pattern Selection

G. Dee and J. S. Langer
Institute for Theoretical Physics, University of California at Santa Barbara, Santa Barbara, California 93106

(Received 6 December 1982)

Pattern selection is discussed in regard to a situation where a stable, nonuniform state of a nonlinear dissipative system propagates into an initially unstable, homogeneous region. The velocity of the propagating front and the wavelength of the pattern formed behind the front are determined by a marginal-stability criterion. The special system studied here has a Lyapunov functional, but the periodic state which propagates is not the one which minimizes the functional.

PACS numbers: 05.70.Ln

Development of a meaningful theory of natural pattern selection requires models of pattern-forming mechanisms that are simple enough to be understood in detail. Real systems are intrinsically complex and exhibit a variety of responses to differing experimental situations. One would like, for example, to predict the periodicities of Rayleigh-Bénard convection patterns,[1] of cellular solidification fronts,[2] cellular flame fronts,[3] etc. The basic difficulty is that the steady-state descriptions of each of these systems admit whole bands of linearly stable states; yet, at least under some conditions, unique states are selected in real experiments. It will ultimately be important to understand whether, or under what circumstances, pattern formation in such systems is an intrinsic property of the systems themselves or, perhaps, depends sensitively on initial configurations, boundaries, or externally imposed perturbations.[4]

A particularly striking example of the pattern-selection problem occurs in the theory of dendritic solidification.[1,5] In this case, it appears that the naturally selected states are those which sit just at the margin of instability. Although qualitative arguments have been advanced in favor of this principle of marginal stability, no systematic derivation has been discovered.

The first group of pattern-forming phenomena mentioned above has the common feature that periodic structures are emerging in translationally symmetric systems. A particular mechanism for pattern formation in such systems, which we shall call "pattern propagation," has some similarity to the dendritic process.[6] Consider an initially structureless system which is "quenched" so that it becomes uniformly unstable against pattern-forming deformations. A perturbation which at first is confined to a small region will grow locally into a well developed pattern—convective rolls, cellular structures, etc.—and this pattern will spread out into the rest of the space. This pattern may spread by propagating at a well-defined velocity, the front of the pattern looking much like the tip of a dendrite which generates an array of side branches behind it as it moves. A picture of such a pattern front is shown in Fig. 1. The crucial questions are the following: What is the speed of the front? What is the wavelength of the pattern which is stabilized behind this front?

An instructive example is a nonlinear diffusion equation which has been much discussed in the literature of mathematical biology.[7-10] We shall

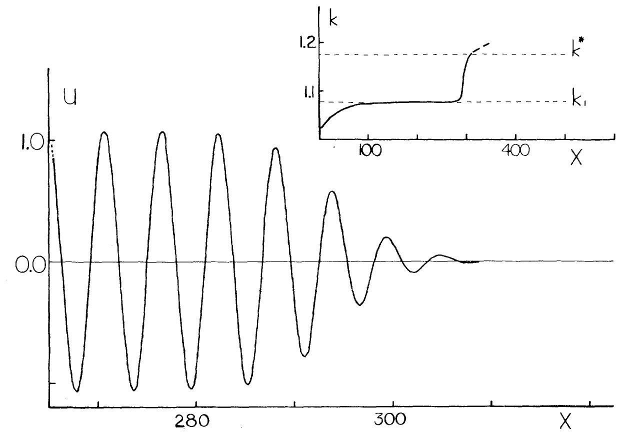

FIG. 1. Front portion of a propagating pattern determined by Eq. (4) for $\epsilon = 0.9$. The oscillatory part of the pattern on the left is stationary in the laboratory frame, and new oscillations arise as the envelope of the pattern moves to the right. Inset: the local wave number k as a function of x for the entire system.

consider at first only a specially symmetric version, viz.

$$\partial v/\partial t = \partial^2 v/\partial x^2 + v - v^3, \qquad (1)$$

which is also well known in the physics literature as a model for phase transformations. Stable states of this system occur at $v = \pm 1$, and an unstable state at $v = 0$. The situation of interest is one in which the unstable state $v = 0$ is transformed into, say, $v = 1$ at a front moving at speed c. This steady-state front must be described by a function $v_c(x')$, $x' = x - ct$, which is a solution of

$$0 = \partial^2 v_c/\partial x'^2 + c\,\partial v_c/\partial x' + v_c - v_c^3 \qquad (2)$$

with the boundary conditions $v_c \to 1$ at $x' \to -\infty$, $v_c \to 0$ at $x' \to +\infty$.

Solutions of (2) exist for all positive values of c. To see this, note that (2) can be interpreted as the mechanical equation of motion for a particle of unit mass whose "displacement" v_c is a function of "time" x'. The particle is undergoing damped motion with damping constant c in a potential $\frac{1}{2}v_c^2 - \frac{1}{4}v_c^4$. The relevant trajectories are those in which the particle starts with zero speed from the potential maximum at $v_c = 1$ and falls to the minimum at $v_c = 0$. For $c > 2$, the motion is overdamped, and the front $v_c(x')$ is monotone decreasing. For the underdamped case, $c < 2$, the forward part of the front oscillates as it approaches $v_c = 0$.

A definitive mathematical discussion of Eq. (1) has been published by Aronson and Weinberger,[10] who prove that all initial states $v(x, t=0)$ which lie within the strip $0 \leq v \leq 1$, which do not vanish everywhere, and which decrease at least as fast as e^{-x} in the $+x$ direction will form propagating fronts with $c = 2$. That is, the basin of attraction for the state v_2 is overwhelmingly bigger than that for any other and, in particular, contains all the physically achievable initial perturbations which are of bounded extent. This, then, is an example of a sharp selection mechanism.

This selected state is marginally stable. The term "stability" is used here in just the same sense as it has been used in the theory of dendritic growth.[5,6] That is, we look in the frame of reference moving with the front and ask whether an initially localized perturbation, observed at a fixed point in that frame, will grow or decay. A perturbation which decays is considered stable even if it generates a growing disturbance, like a sidebranch, which moves away from its point of origin near the tip. The detailed analysis of the stability spectrum for Eq. (1) will be published elsewhere. For present purposes, we shall use an intuitively appealing but oversimplified picture which we shall present in such a way that it can be applied to a wider class of models than that described in Eq. (1).

Consider a localized perturbation imposed on an otherwise uniform unstable system. For small disturbances, we can linearize the equation of motion, make a Fourier transformation, and obtain a dispersion relation for the amplification rate ω as a function of wave number k. After a long time t, and at a large distance x away from the initial disturbance, the perturbation will have the form $\exp[ik^*x + \omega(k^*)t]$, where k^* is the point of stationary phase in the complex k plane. If we observe this perturbation at a moving position $x = ct$, then we should be able to choose c large enough that we outrun the perturbation, that is, we should see a decaying exponential in time. In this sense, c is a stabilizing parameter. The marginal-stability hypothesis is simply the conjecture that the natural velocity of the pattern front, c^*, is that for which this exponential neither grows nor decays. Therefore, c^* should be obtained by solving

$$ic^* + d\omega/dk^* = 0; \quad \mathrm{Re}[ic^*k^* + \omega(k^*)] = 0. \qquad (3)$$

Note that this analysis works correctly to give $c^* = 2$ when $\omega = 1 - k^2$ as in the case of Eq. (1).

A much more interesting case to consider is the equation[11,12]

$$\partial u/\partial t = [\epsilon - (\partial^2/\partial x^2 + 1)^2]u - u^3, \qquad (4)$$

Here, ϵ is a control parameter introduced in such a way that the state $u = 0$ becomes unstable when ϵ becomes positive. The restabilized stationary solutions of (4), for any fixed ϵ in the range $0 \leq \epsilon \leq 1$, are periodic functions with fundamental wave numbers k occurring in bands of finite width in the neighborhood of $k \sim 1$. For small ϵ, (4) can be approximated by an amplitude equation by writing

$$u(x, t) \cong (\tfrac{1}{3}\epsilon)^{1/2} W(X, \tau) e^{ix} + (\tfrac{1}{3}\epsilon)^{1/2} W^*(X, \tau) e^{-ix}, \qquad (5)$$

where $X = \frac{1}{2}\epsilon^{1/2}x$ and $\tau = \epsilon t$. Then the amplitude W satisfies

$$\partial W/\partial \tau = \partial^2 W/\partial X^2 + W - |W|^2 W + O(\epsilon^{1/2}). \qquad (6)$$

This is the same as (1) except that W may be complex. There are strong reasons for believing that localized initial conditions must evolve according to (6) into propagating states in which the phase of W becomes very nearly constant, so that the theorems of Aronson and Weinberger are applicable. This leads to the prediction that, to lowest order in ϵ, $c^* = 4\epsilon^{1/2}$. The correspondence between (4) and (1) via the amplitude Eq. (6) does not prove, but makes it plausible, that pattern propagation in this more complicated model occurs at a sharply defined speed.

The simplified marginal-stability theory summarized in (3) may be used to compute the function $c^*(\epsilon)$ throughout the physically interesting range ($0 < \epsilon < 1$) by setting $\omega(k) = \epsilon - (k^2 - 1)^2$. The results are shown in Fig. 2. This theory may also be used to predict the wavelength of the pattern which emerges behind the front. In doing this, it is helpful to look at the wave form in Fig. 1, which is a computer-generated solution of Eq. (4) for $\epsilon = 0.9$. The fully developed pattern has a wave number k_1 and is stationary in the fixed frame of reference. Alternatively, the pattern may be visualized as moving with velocity $-c^*$ in the frame of reference in which its envelope is at rest. Ahead of the front, the pattern is predicted by (3) to have a wave number $\mathrm{Re}k^*$ and to be oscillating at a fixed point in the moving frame, with an angular frequency

$$\Omega \equiv \mathrm{Im}[ik^*c^* + \omega(k^*)]. \qquad (7)$$

This frequency may be interpreted as a flux of nodes moving in the $-x$ direction relative to the envelope. As long as nodes are not created or destroyed when they pass through the front of the pattern, this flux must be conserved and must be equal to $k_1 c^*$ in the bulk. Thus, we predict

$$k_1 = \Omega/c^*. \qquad (8)$$

For example, for $\epsilon = 0.9$ we find $c^*/\epsilon^{1/2} = 4.546$, $\mathrm{Re}k^* = 1.1758$, $\Omega = 4.643$, and $k_1 = 1.076$.

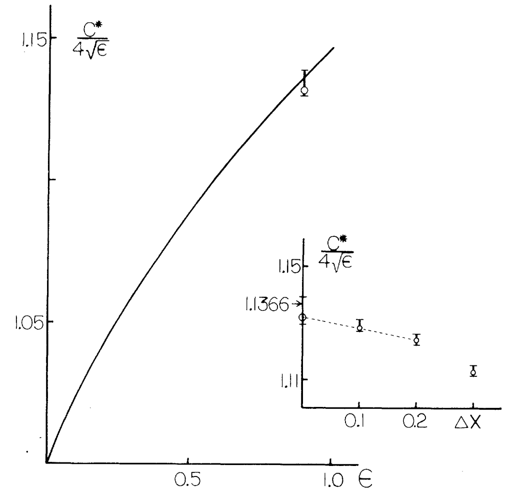

FIG. 2. Scaled velocity $c^*/4\sqrt{\epsilon}$ as a function of the control parameter ϵ. Inset: the extrapolation to vanishing grid spacing used in estimating c^* for $\epsilon = 0.9$.

These predictions for c^* and k_1 have been checked by direct numerical solution of Eq. (4). Our procedure is straightforward; we start with a localized perturbation and watch it evolve as shown in Fig. 1. Our results for the propagation speed are generally consistent with the predicted $c^*(\epsilon)$ shown in Fig. 2. The inset in that figure shows measured values of c for $\epsilon = 0.9$ at three different grid sizes, with our proposed extrapolation indicated by the dashed lines. The error bars indicate only the scatter in our data and not any estimate of systematic error, except that the bar for the point at the smallest grid size is skewed because the speed of the front seemed still to be increasing slowly at the end of this run.

The inset in Fig. 1 shows the local wave number k as a function of x for the entire system whose front is shown in the main part of that figure. Note that the wave number in the body of the pattern has settled down accurately to the predicted k_1. Ahead of the front, k passes through $\mathrm{Re}k^*$ and the amplitude of the wave form decreases rapidly. The variation of k near $x = 0$ is a vestige of the initial transient which has been trapped behind the front and is slowly disappearing diffusively. One of the reasons that we must use long computer runs to evaluate c is that we must allow enough time for the front to become well separated from this transient.

The results of this investigation are encouraging in the sense that we seem to have found a sharp selection mechanism which is an intrinsic property of the system, independent of detailed initial conditions, boundary effects, or external perturbations. On the other hand, these results are discouraging to hopes of finding universal selection criteria. In the first place, the mechanism described here depends only on linear properties of the unstable part of the system, and we know that this cannot be true in general. For example, the addition of a term of the form v^2 to the right-hand side of (1) invalidates the simplified version of the marginal-stability hypothesis summarized in (3). It turns out that marginal stability is still correct, but the instability which controls the speed of propagation is a localized deformation of the front which appears only in the fully nonlinear analysis. We do not know under what circumstances such a nonlinear effect might occur in a pattern-forming model like (4), but we see no reason to exclude this possibility.

An even more serious point to recognize is that there exists a Lyapunov function $F\{u\}$ for Eq. (4), and that the selected wave number k_1 is not the one which minimizes F. That is, given reasonable boundary conditions of the kind used here, we can write (4) in the form

$$\partial u/\partial t = -\delta F/\delta u,$$
$$F = \int dx \left[\frac{1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 - \left(\frac{\partial u}{\partial x}\right)^2 + \tfrac{1}{2}(1-\epsilon)u^2 + \tfrac{1}{4}u^4\right], \tag{9}$$

and note that $dF/dt \leq 0$. For $\epsilon = 0.9$, F has its absolute minimum at the stationary state with fundamental wave number $k = 0.998$, which differs from $k_1 = 1.076$ by an amount which is well beyond our numerical uncertainty. Thus, patterns which form by propagation will not be the same as those which are most stable in the presence of, say, thermal fluctuations.

Finally, we remark that both k_1 and Rek^* lie well within the band of stable stationary solutions of (4). The Eckhaus instability, where the phase-diffusion constant for the bulk pattern vanishes, occurs at $k = 1.25$ for $\epsilon = 0.9$. Thus, the selected wave number in this case definitely does not occur near a marginal instability of the bulk pattern, as has been suggested in another context.[13]

This work was supported by the U. S. Department of Energy, under Contract No. DE-AM03-765F00034 and by the National Science Foundation under Grant No. PHY77-27084. The research was carried out as part of an Institute for Theoretical Physics program on nonequilibrium phenomena, and many of the early ideas were developed in collaboration with E. Ben-Jacob, H. Brand, and L. Kramer. The authors are also indebted to R. Ball, J.-P. Eckmann, P. C. Hohenberg, and G. Schön for important contributions.

[1]A. Schluter, D. Lortz, and F. Busse, J. Fluid Mech. 23, 129 (1965); A. C. Newell and J. C. Whitehead, J. Fluid Mech. 38, 279 (1969).

[2]J. S. Langer, Rev. Mod. Phys. 52, 1 (1980).

[3]G. I. Sivashinsky, "Instabilities, Pattern Formation, and Turbulence in Flames," to be published.

[4]L. Kramer, E. Ben-Jacob, H. Brand, and M. C. Cross, Phys. Rev. Lett. 49, 1891 (1982).

[5]J. S. Langer and H. Müller-Krumbhaar, Acta Metall. 26, 1681, 1689, 1697 (1978).

[6]J. S. Langer and H. Müller-Krumbhaar, Phys. Rev. A 27, 499 (1982). Pattern propagation has also been observed in numerical simulations by Kramer *et al.*, Ref. 4, who have studied both the dendritic model and coupled reaction-diffusion systems.

[7]R. A. Fisher, Ann. Eugenics 7, 355 (1937).

[8]A. Kolmogoroff, I. Petrovsky, and N. Pisconnoff, Bull. Univ. Moscow, Ser. Internat., Sec. A 1, 1 (1937).

[9]D. G. Aronson and H. F. Weinberger, in *Partial Differential Equations*, Lecture Notes in Mathematics Vol. 446, edited by J. A. Goldstein (Springer, New York, 1975), p. 5.

[10]D. G. Aronson and H. F. Weinberger, Adv. Math. 30, 33 (1978).

[11]J. Swift and P. C. Hohenberg, Phys. Rev. A 15, 319 (1977).

[12]Y. Pomeau and P. Manneville, J. Phys. Lett. 40, 609 (1979). See also M. C. Cross, P. G. Daniels, P. C. Hohenberg, and E. D. Siggia, to be published.

[13]J. S. Langer, Phys. Rev. Lett. 44, 1023 (1980).