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Critical Velocity for a Self-Sustaining Vortex Tangle in Superfluid Helium

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A recent treatment of superfluid turbulence is extended to the study of turbulence in a channel. It is found that as the flow velocity is reduced, a critical velocity is reached such that the rate at which new vortex singularities are created by line-line reconnections becomes insufficient to balance the loss of vortices at the channel walls, and the vortex tangle ceases to be topologically self-sustaining. Comparison with experiment indicates that this approach provides a reasonable explanation of observed critical velocities.

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One of the more interesting properties of superfluid ^{4}He is that it can flow without dissipation. It is well known, however, that this is true only up to some maximum flow velocity, above which the fluid undergoes a transition to a kind of turbulent state in which it becomes filled with a tangle of quantized vortex lines. Numerous elegant experiments have established the basic properties of this instability,[1-8] yet its origin has remained an enduring mystery.[9] In this paper, a recently developed method for treating superfluid turbulence[10] is used to investigate the vortex tangle in a channel, with the particular aim of finding out what happens to the tangle as the average flow velocity is reduced. This approach yields a simple qualitative explanation of the transition, and allows one to calculate its properties.

If the curve $\vec{s}=\vec{s}(\xi,t)$ specifies the instantaneous configuration of vortex-line singularities which make up the tangle, the instantaneous motion of the line with respect to the local average superfluid velocity is to a good approximation given by[11]

$$\partial\vec{s}_0/\partial t = \vec{s}_0'\times\vec{s}_0'' + \alpha\vec{s}_0'\times(\vec{v}_0 - \vec{s}_0'\times\vec{s}_0''), \qquad (1)$$

where $\vec{s}_0'$ is the vector tangent and $\vec{s}_0''$ the vector curvature at the point in question, $\vec{v}_0=\vec{v}_{n0}-\vec{v}_{s0}$ is the local relative velocity between the normal fluid and the superfluid, and α is the coefficient which measures the force exerted by the normal fluid on the vortex line.[12] The zero subscripts signify that Eq. (1) is given in reduced units, such that length is expressed in terms of the characteristic dimension D, velocities in terms of the corresponding Feynman velocity β/D,[13] and time in terms of D^2/β. Equation (1) breaks down on the relatively infrequent occasions when vortex lines try to cross. It is assumed[14] that whenever this occurs, the vortex singularities will undergo a topology-changing reconnection. As was pointed out previously,[10] this new idea is of central importance in developing an understanding of superfluid turbulence, since it is through this process that new vortex singularities are generated.

Equation (1) plus the reconnection *Ansatz* give a complete, although slightly idealized, prescription for calculating the time development of the vortex tangle. The combination of Eq. (1) acting on randomly curving singularities and the occasional, discrete reconnection events leads to a highly nonlinear problem, which, however, turns out to be well suited for numerical simulations. Thus it was shown in Ref. 10 that the properties of homogeneous steady-state turbulence can be calculated directly by implementing this prescription on the computer.

The model is easily augmented to include the effect of real boundaries by noting that vortex lines which approach such a boundary sufficiently closely will undergo a line-surface reconnection[15] of the kind shown in Fig. 1. With this simple addition, the powerful yet completely transparent approach outlined above can be applied to study superfluid turbulence in a channel. In the actual calculations, a sample volume consisting of a suitably long section of the channel is filled with an arbitrary initial vortex configuration and subjected to a uniform driving velocity v_0 along the channel flow direction. The development of this configuration as it interacts with the channel walls as well as itself is then followed by explicit calculation.[16] It is found that the vortex tangle in the channel reaches a well-defined steady state about which it fluctuates in a random manner. At lower values of v_0, the average steady-state line-length density L_0 falls increasingly below the corresponding homogeneous-turbulence value, and at a well-defined critical velocity v_{0c} it drops discontinuously to zero. *Thus, the existence of a critical velocity below which the vortex tangle ceases to be topologically self-sustaining follows directly from Eq. (1) plus the reconnection Ansatz, which in turn are derivable from simple vortex dynamics.* From the calculations it is clear that the tangle ceases to be self-sustaining when the line-length density drops to a level such that the rate at which new growth loops are created by the reconnection process cannot keep up with the annihilation of vortex lines at the channel walls.

In contrast to the calculations on homogeneous turbulence,[10] where dimensional scaling arguments obviate much of the computational labor, the calculation of $L_0(v_0)$ is quite arduous. Nevertheless, sufficient calculations have been performed to permit comparison with various experimentally established features of the critical velocity problem. One may note at the outset that if $L_0(v_0)$ has been calculated for a particular channel geometry, the scaled-out solution for a channel of characteristic dimension D is

$$L(v) = L_0(vD/\beta)/D^2. \tag{2}$$

It follows that the critical velocity must scale as

$$v_c \cong v_{0c}(\kappa/4\pi D)\ln(c_1/s''a_0), \tag{3}$$

where β has been inserted explicitly. The relation (3), versions of which (with v_{0c} omitted) occur often in discussions of the critical velocity problem, is seen to be an essentially trivial result. The real problem lies in determining v_{0c}.

Figure 2 shows the steady-state vortex-line density L_0 as a function of v_0, calculated for $\alpha = 0.1$ ($T \cong 1.6$ K) in a square channel with smooth walls, with a driving velocity which is uniform over the entire channel. Note that L_0 is the average of a fluctuating quantity, and furthermore is an average over the channel cross section. Although such curves also provide other information which may be compared with experiment, the initial interest is in the finite critical velocity that they exhibit.

The calculations leading to Fig. 2 were repeated with $\alpha = 0.03$ ($T \cong 1.25$ K) and $\alpha = 0.3$ ($T \cong 2.0$ K). The final results for the critical velocity are given by the open circles in Fig. 3. For comparison, the points show the experimental values of critical velocity obtained by Childers and Tough[5] in a

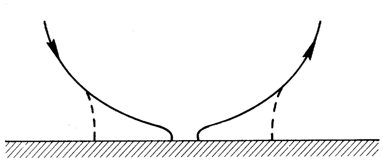

FIG. 1. Solid line, schematic representation of a line-surface reconnection as it would occur naturally. Dashed line, reconnection made artificially far out to inhibit surface-enhanced loop production.

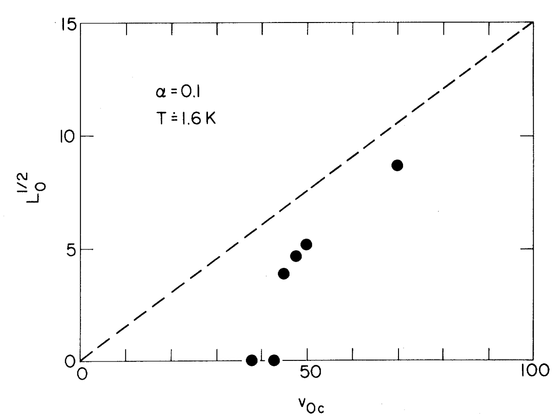

FIG. 2. Computed values of the steady-state line-length density in a square channel for various values of the driving velocity. The dashed line gives the homogeneous-turbulence results.

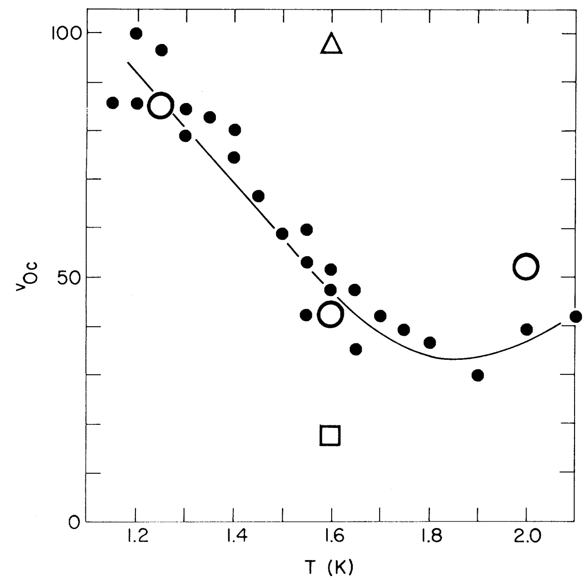

FIG. 3. Open circles, computed values of the critical velocity in a square, smooth channel. Points are the measured values of Ref. 5, in reduced units. The line is drawn to guide the eye. The open square shows the effect of going from a square channel to a parallel-plate geometry. The open triangle shows the effect of inhibiting the surface enhancement.

circular channel, with the normal and superfluid components in counter flow. At the present rudimentary stage of the theory, the high degree of numerical agreement seen here should not be taken too seriously. Other experiments give critical velocities differing by factors of as much as 2 or 3. Also, as discussed below, the calculations themselves turn out to be sensitive to factors such as pinning and nonuniform driving fields, the accurate treatment of which requires further refinements of the theory. Nevertheless, it is clear that both the approximate magnitude and the temperature dependence of the critical velocity are successfully accounted for.

Ladner and Tough[7] have established that the critical velocity is affected not only by the characteristic dimension of the channel, but also by its geometry. They find that a channel with a 10:1 rectangular cross section exhibits a v_{0c} which is about a factor of 2.5 less than that found for a 1:1 (square) cross section. To test the theory in this regard, the critical velocity for flow between parallel plates at $\alpha = 0.1$ ($T \cong 1.6$ K) was calculated.[17] Presumably, the parallel-plate geometry should approximate the 10:1 rectangular channel. The calculated v_{0c}, indicated in Fig. 3 by a hollow square, is found to drop by a factor of ~2.5, in perfect agreement with experiment. Thus the observed geometry dependence of v_{0c} also seems to be accounted for by the theory.

Examination of the calculational output uncovered the surprising fact that near v_{0c} the turbulence is not in fact maintained by new loops created in the interior of the channel, but primarily by loops created at the walls. This *surface enhancement* of the loop-creating process arises when a surface reconnection occurs (Fig. 1). The resulting large self-induced velocities cause the end of the vortex to move rapidly around on the surface, leading to an enhanced probability of line-line reconnections near there. When this surface enhancement is artificially inhibited, e.g., by making the surface reconnections as shown by the dashed line in Fig. 1, the critical velocity increases by a factor of 2 to 4. This higher critical velocity may be interpreted roughly as the velocity at which the turbulence becomes self-sustaining as a result of "bulk" loop creation in the channel interior.

The higher critical velocity should be observable only in experimental situations where the surface enhancement is inhibited. One expects, for example, that loop production near a wall would tend to be decreased when the wall offers many pinning sites, as is the case for metal walls. A second inhibiting mechanism should operate in situations such as counterflow, where the turbulence is driven mainly by the normal-fluid velocity. Since the normal-fluid velocity has to go to zero at the walls, loop growth there will be decreased.

In light of these expectations, it is very interesting to note that in some experiments,[1] a well-defined kink or discontinuity in the $L_0(v_0)$ curve is observed at a "secondary" critical velocity typically a few times larger than the "primary" v_{0c}. It would seem natural to identify this secondary velocity with the onset of a turbulent state maintained by the creation of loops in the interior of the channel. The experimentally observed trends lend weight to this conjecture: The secondary feature appears as a very pronounced discontinuity for counterflow in metal tubes, it appears as a weak kink for counterflow in smooth glass tubes, and it is absent for pure superflow in glass tubes. It must be noted, however, that the artificial method of inhibiting the surface enhancement used so far in the calculations merely produces a curve like the one in Fig. 2 moved to higher velocities, not one which shows two critical velocities. Clearly a more realistic treatment of pinning effects and of the nonuniform normal-fluid field will be required to sort out in detail the various kinds of transition behavior which have been observed.

In summary, a description of superfluid turbulence which emphasizes the importance of line-line and line-surface reconnections leads to a very natural solution of the critical-velocity problem. The values of the calculated critical velocity as a function of channel size, temperature, and aspect ratio are all in substantial agreement with experiment. The threshold behavior is found to be sensitive to various surface effects, in a manner strongly reminiscent of laboratory observations. Further refinements of the calculation are desirable, in particular with respect to pinning, nonuniform driving fields, and end effects. Perhaps more importantly, the question of how the turbulent state is initiated, presumably from some original distribution of pinned vortices, as suggested by Vinen,[9] is not yet understood. This question, which is logically and experimentally distinct from the critical-velocity problem, should be amenable to the same kind of treatment.

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[13]$\beta = (\kappa/4\pi)\ln(c_1/s'' a_0)$ where κ is the quantum of circulation, $a_0 \sim 10^{-8}$ cm is the vortex cutoff parameter, and c_1 is a constant of order 1.

[14]In Ref. 3 and the present paper, the reconnection hypothesis appears as an *Ansatz*. It can, however, be *shown* to be true by doing a full, nonlocal treatment of the line-line interaction. This will be discussed in a later paper.

[15]Again, this can be shown to be true by looking at the line-surface interaction in detail.

[16]Reentrant boundary conditions are applied to the ends of the sample section. This is equivalent to filling the channel with an infinite repetition of the sample section.

[17]One can go from the square channel to the parallel-plate problem by replacing one pair of channel walls by surfaces obeying reentrant boundary conditions.