ma, private communication.

[5]G. Ahlers, D. S. Cannell, and M. A. Dominguez Lerma, Phys. Rev. Lett. 49, 368 (1982); Figure 2 shows a plot of the time-averaged vortex-pair width distribution. We observe a similarly nonuniform distribution in our $m=2$ flow.

[6]A quantitatively reliable measurement of mode amplitude distribution, from which vortex spacings can be obtained, is a difficult task, with many technical problems such as errors introduced by angles of viewing and illumination. We are attempting such a measurement.

[7]T. Brooke Benjamin, Proc. Roy. Soc. London, Ser. A 359, 27 (1978).

[8]K. Park, G. L. Crawford, and R. J. Donnelly, Phys. Rev. Lett. 47, 1448 (1981).

Characterization of Strange Attractors

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A new measure of strange attractors is introduced which offers a practical algorithm to determine their character from the time series of a single observable. The relation of this new measure to fractal dimension and information-theoretic entropy is discussed.

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Dissipative dynamical systems which exhibit chaotic behavior often have an attractor in phase space which is *strange*.[1-3] Strange attractors are typically characterized by fractal dimensionality[4] D which is smaller than the number of degrees of freedom F, $D<F$. So far, this fractal (or Hausdorff) dimension has been the most commonly used measure of the "strangeness" of attractors.[5-10] Several attempts to compute this number directly from box-counting algorithms, which stem from the definition of this dimensionality, have been presented.[7-10] It turns out that it is very difficult[10] to compute D whenever $D>2$. Most importantly, the use of a single time series of any observable to extract this measure of the attractor has been found to be impractical for dynamical systems which possess attractors whose $D>2$.[10] An important question is then how to analyze experimental signals. In this Letter we suggest a different measure for the strangeness of attractors, a measure which can be easily obtained from any time series without resorting to Poincaré maps,[3] and which is closely related to the fractal dimension. We shall attempt to argue that in fact this measure is more relevant in many cases than D itself.

The measure is obtained by considering correlations between points of a long-time series on the attractor. Denote the N points of such a long-time series by $\{\vec{X}_i\}_{i=1}{}^N \equiv \{\vec{X}(t+i\tau)\}_{i=1}{}^N$, where τ is an arbitrary but fixed time increment. The definition of the correlation *integral* is

$$C(r) \equiv \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \theta(r - |\vec{X}_i - \vec{X}_j|)$$
$$\equiv \int_0^r d^d r' \, c(\vec{r}'), \tag{1}$$

where $\theta(x)$ is the Heaviside function and $c(\vec{r})$ is the standard correlation function. The main point of this paper is that $C(r)$ behaves as a power of r for small r:

$$C(r) \propto r^{\nu}. \tag{2}$$

Moreover, the exponent ν is closely related to D as well as to a properly defined entropy which is discussed below. Before continuing the analysis we show in Fig. 1 two examples of the behavior (2).

Shown are the logarithms of the correlation integrals for the Hénon map[11] [Fig. 1(a)] and the Lorenz model[1] [Fig. 1(b)] as a function of log$r$. Equally convincing power laws were obtained for the Kaplan-Yorke map,[5] the Rabinovich-Fabrikant[12] equations, and the logistic map[13] at the onset of chaos (see Table I). For the Zaslavskii map[15] no clear power law is obtained even for longer runs.

One sees from Table I that ν is in all cases very close to D (the errors quoted are "educated guesses"), but is never greater than D (with the exception of Zaslavskii's map where no good power law is seen). Below, we shall argue that

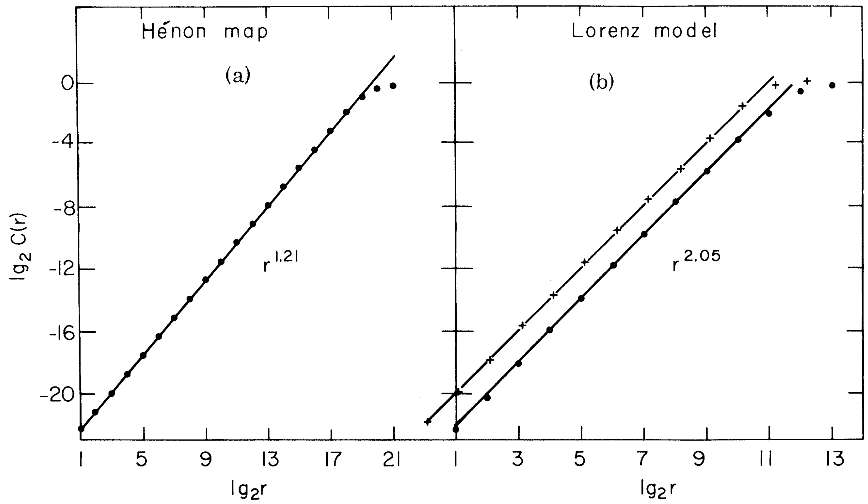

FIG. 1. Correlation integrals for (a) Hénon map and (b) Lorenz model on doubly logarithmic scales. In (b) the upper line was computed from a single variable time series. In both panels the scale of r is arbitrary.

the value $\nu = 1.21$ obtained for the Hénon map is wrong as a result of systematic errors. An improved method yields indeed $\nu = 1.25 \pm 0.02$, such that $\nu = D$ within the errors. We now discuss the relation between various measures of the strangeness of attractors.

Consider a coverage of the attractor by hypercubes of edge length l. If the attractor is a fractal, then the number $M(l)$ of cubes that contain a piece of the attractor is[4]

$$M(l) \sim l^{-D}. \tag{3}$$

Denote now by μ_i ($i = 1, 2 \ldots$) the number of points from the set $\{\vec{X}_i\}_{i=1}^N$ which are in the ith nonempty cube. Up to a factor of $O(1)$ (i.e., the number of nearest-neighbor cubes) we can write

$$C(l) \sim \frac{1}{N^2} \sum_{i=1}^{M(l)} \mu_i^{\,2} = \frac{M(l)}{N^2} \langle \mu^2 \rangle, \tag{4}$$

TABLE I. Maps used in evidence, with values of corresponding parameters.

	ν	No. of iterations, time increment τ	D	σ
Hénon map,	1.21 ± 0.01[d]	15 000	1.26[g]	...
$a = 1.4$, $b = 0.3$	1.25 ± 0.02[e]	20 000		
Kaplan-Yorke map,				
$\alpha = 0.2$	1.42 ± 0.02	15 000	1.431[g]	...
Logistic equation,	0.500 ± 0.005			
$b = 3.569\,945\,6\ldots$	$0.4926 < \nu < 0.5024$[f]	25 000	0.538[h]	0.517 097 6[i]
Lorenz equation[a]	2.05 ± 0.01	15 000; $\tau = 0.25$	2.06 ± 0.01[g]	...
Rabinovich-				
Fabrikant equation[b]	2.18 ± 0.01	15 000; $\tau = 0.25$	...	...
Zaslavskii map[c]	(~ 1.5)	25 000	1.39[g]	...

[a] Parameters as in Refs. 10 and 6.
[b] Parameters as in Sec. 3 of Ref. 12.
[c] Parameters as in Ref. 7.
[d] From Eqs. (1) and (2).
[e] From single-variable time series, with $f = 3$.
[f] Exact analytic bounds (Ref. 14).
[g] Ref. 7.
[h] Ref. 9.
[i] Ref. 14.

where angular brackets denote an average over all occupied cells.

By the Schwartz inequality,

$$C(l) \geq \frac{M(l)}{N^2} \langle\mu\rangle^2 = \frac{1}{N^2 M(l)} \Big[\sum_{i=1}^{M(l)} \mu_i\Big]^2 = 1/M(l) \sim l^D, \tag{5}$$

where $\sum \mu_i = N$ and Eq. (3) have been used. From this it follows that

$$\nu \leq D. \tag{6}$$

To gain further understanding of the relation between these two measures of strangeness we introduce a third one, information-theoretic entropy.[16] This is the minimal information needed to pin down a point on the attractor with precision l:

$$S(l) = -\sum_{i=1}^{M(l)} p_i \ln p_i, \tag{7}$$

where p_i is the probability for a point to fall in the ith cube (for $N \to \infty$, $p_i = \mu_i/N$). For a *uniform* coverage [i.e., $p_i = 1/M(l)$] the entropy is $S^0(l) = \ln M(l) = \text{const} - D \ln l$. In the general case $S(l) < S^0(l)$. If we adopt the *Ansatz*

$$S(l) = S_0 - \sigma \ln l \tag{8}$$

(with σ called "information dimension" by Farmer, and "dimension" by Renyi[16]), we are led to the inequality

$$\sigma \leq D. \tag{9}$$

Finally we want to prove that $\nu \leq \sigma$, thus estimating σ from above and below. A sketch of the proof is as follows. Consider two nested coverings with cubes of lengths l and $2l$, respectively. Evidently, $M(l) = 2^D M(2l)$. Define p_i as the probability for a point to fall in cube i of the fine covering and P_j the probability for it to fall in the jth cube of the coarser covering, $P_j = \sum_{i\in j} p_i$. Define now ω_i by $p_i = \omega_i P_j$. Evidently $\sum_{i\in j}\omega_i = 1$.

According to Eq. (4) the correlation integral $C(l)$, up to a factor of $O(1)$, is

$$C(l) \sim \sum_{i=1}^{M(l)} p_i^{\,2} = \sum_{j=1}^{M(2l)} P_j^{\,2} \sum_{i\in j} \omega_i^{\,2}.$$

If we assume now that ω_i is independent of j, we can write $C(l)/C(2l) = \langle\omega^2\rangle/\langle\omega\rangle$. On the other hand we consider

$$S(2l) - S(l) = \sum_j P_j \sum_{i=j} \omega_i \ln\omega_i = \langle\omega \ln\omega\rangle/\langle\omega\rangle.$$

Defining the quantity $W = \omega/\langle\omega\rangle$ we can employ the inequality[17,18] $\langle W^2\rangle \geq \exp[\langle W \ln W\rangle]$ to prove the inequality $\nu \leq \sigma$. We thus find that, combining ν and D *together*, we can have an excellent estimate of the information content of a strange attractor via the set of inequalities

$$\nu \leq \sigma \leq D. \tag{10}$$

It is important to stress that when the covering of the attractor is uniform, the equalities in Eq. (10) are realized.[18,19] The fact that $\nu \neq D$ in the logistic map shows that in this case the coverage is not uniform. Certain neighborhoods have higher "seniority" in the sense that they are visited more often than others. The fractal dimension is ignorant of the seniority. It has to do only with the geometrical structure of the attractor. Regions of the attractor which are rarely visited contribute to D with equal weight as regions of high visiting rate. The correlation integral (and the entropy) are, however, sensitive to this effect. In this sense ν may be a more relevant measure of the attractor than D because it is sensitive to the *dynamical* process of coverage of the attractor. The difference between ν and D gives a measure of the importance of different seniority of different neighborhoods.

Although the data of Table I were obtained using ~20 000 points, convergence in all cases expect Zaslavskii's map was already apparent with a few thousand points. (This should be compared with 200 000 points needed to obtain convergence of the box-counting algorithm used to compute D in the case of the Hénon map, and the lack of convergence with the same number of points in the case of the Lorenz model.)

A variant of our method, inspired by Refs. 20 and 21, consists of measuring instead of $\vec{X}_i$ only one component, say X_i. A new f-dimensional phase space is then constructed by using vectors

$$\vec{\xi}_i = (X_i, X_{i+\tau}, X_{i+2\tau}, \ldots, X_{i+f\tau}),$$

which are then inserted in Eq. (1) instead of the $\vec{X}_i$. An example of the results obtained from this procedure with $f = 3$ is shown in Fig. 1(b) for the Lorenz model. The + marks were obtained by following the x variable only. The power law is still satisfactory. The value of ν calculated from the one-variable data is 2.06 ± 0.02. From an experimental point of view this procedure is much preferable. In fact, it allows a consistent determination of ν in a high-dimensional dynamical system, as will be shown in a forthcoming publication.[14] Also, it allows elimination of systematic errors due to corrections to scaling, by choosing the "embedding dimension" f larger than strictly necessary. Choosing $f=3$ for the Hénon map we get, e.g., the value $\nu = 1.25 \pm 0.02$ quoted above. More details will be presented in Ref. 14.

To summarize, we have introduced the exponent ν of the power-law dependence of the correlation integral as a new measure for the strangeness of attractors.[22] The value of this exponent can be obtained from a time series of one or more variables. The computation is relatively easy and converges rapidly. The relation of the exponent ν to the fractal dimension and information entropy have been discussed. If the attractor is visited by the trajectory uniformly, all these measures coalesce. Otherwise, we attempted to argue that ν might be more dynamically relevant than D. It is our hope that this new characteristic exponent would be actually measured in experimental systems whose dynamics is governed by strange attractors.

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[a]Permanent address: Physics Department, University of Wuppertal, Wuppertal, West Germany.

[1]E. N. Lorenz, J. Atmos. Sci. 20, 130 (1963).

[2]D. Ruelle and F. Takens, Commun. Math. Phys. 20, 167 (1971).

[3]E. Ott, Rev. Mod. Phys. 53, 655 (1981).

[4]B. B. Mandelbrot, *Fractals—Form, Chance and Dimension* (Freeman, San Francisco, 1977).

[5]J. C. Kaplan and J. A. Yorke, in *Functional Differential Equations and Approximations of Fixed Points*, edited by H.-O. Peitgen and H.-O. Walther, Lecture Notes in Mathematics Vol. 730 (Springer, Berlin, 1979).

[6]H. Mori, Prog. Theor. Phys. 63, 1044 (1980).

[7]D. A. Russel, J. D. Hanson, and E. Ott, Phys. Rev. Lett. 45, 1175 (1980).

[8]H. Froehling, J. P. Crutchfield, D. Farmer, N. H. Packard, and R. Shaw, Physica (Utrecht) 3D, 605 (1981).

[9]P. Grassberger, J. Stat. Phys. 26, 173 (1981).

[10]H. S. Greenside, A. Wolf, J. Swift, and T. Pignataro, Phys. Rev. A 25, 3453 (1982).

[11]M. Hénon, Commun. Math. Phys. 50, 69 (1976).

[12]M. I. Rabinovich and A. L. Fabrikant, Zh. Eksp. Teor. Fiz. 77, 617 (1979) [Sov. Phys. JETP 50, 311 (1979)].

[13]R. M. May, Nature 261, 459 (1976).

[14]P. Grassberger and I. Procaccia, to be published.

[15]G. M. Zaslavskii, Phys. Lett. 69A, 145 (1978).

[16]J. Balatoni and A. Renyi, Publ. Math. Inst. Hung. Acad. Sci. 1, 9 (1956), and in *Selected Papers of A. Renyi, Vol. 1* (Academiai Budapest, Budapest, 1976); J. D. Farmer, Physica (Utrecht) 4D, 366 (1982).

[17]W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1971), Vol. 2, 2nd ed.

[18]B. B. Mandelbrot, in *Turbulence and Navier-Stokes Equations*, edited by R. Temam, Lecture Notes in Mathematics, Vol. 565 (Springer, Berlin, 1975).

[19]T. A. Witten, Jr., and L. M. Sander, Phys. Rev. Lett. 47, 1400 (1981).

[20]N. H. Packard, J. P. Crutchfield, J. D. Farmer, and R. S. Shaw, Phys. Rev. Lett. 45, 712 (1980).

[21]F. Takens, in *Proceedings of the Symposium on Dynamical Systems and Turbulence, University of Warwick, 1979–1980*, edited by D. A. Rand and L. S. Young (Springer, Berlin, 1981).

[22]After this paper had been submitted for publication we learned about recent reports by L. S. Young and J. D. Farmer, F. Ott, and J. A. Yorke in which smaller measures are discussed. We thank Dr. Farmer for bringing these papers to our attention.