Hadron Spectrum: Evidence for Size Problems on the Lattice

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The question is raised whether sensible light-hadron spectroscopy can be done on the small lattices used in recent analyses. Straightforward arguments together with the results obtained from a high-order hopping-parameter expansion and from quantum mechanical studies for heavy quarks indicate an answer: no.

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It is difficult to find a model in the literature which would give a grossly incorrect hadron spectrum. Models based on completely different physical pictures give similar and reasonable results. The aim of the considerable effort invested into lattice calculations recently[1-10] is to answer the basic question: To what extent is QCD the theory of strong interactions? The purpose is not to derive another set of "reasonable numbers" under uncontrollable approximations or physically unacceptable conditions.

Every lattice calculation carries systematical errors related to the short-distance resolution and to the long-distance cut. A resolution of ~1 fm for instance is clearly not enough for the description of the hadronic world. The bad resolution is one of the reasons that the hadron spectrum in the strong-coupling limit is not considered as being relevant to the basic question above (although the numbers are quite reasonable). Similarly, a minimal required size for the spacelike lattice is determined by the size of the investigated bound state at the given coupling constant.

The long-distance conditions are basically inconsistent in the hadron spectrum calculations performed until now. The lattice distance used in these works implies a box size of ~0.73 fm (Refs. 1, 6, and 9), $\lesssim 0.88$ fm (Ref. 8) at $1/g^2 = 1.0$, or ~1.2 fm (Ref. 7) at $1/g^2 = 0.95$.[11] The graphs of a tenth-order hopping-parameter expansion cover a similarly small region.[5] On the other hand, the expected diameter of a light hadron is of the order of ~2 fm.

Either the lattice hadron is very small, signaling that the point $1/g^2 = 1.0$ is in the strong-coupling regime, or, if the hadron has the expected size, the lattice is far too small to accommodate it.

The results we present here, which are based mainly on a high-order (24th-order) hopping-parameter expansion, are consistent with the second possibility. An analytic study of these rather long series is used to identify the position of singularities on the complex energy plane as a function of the quark masses. For heavy quarks the result is compared with a nonrelativistic quantum mechanical calculation on the lattice. They all imply that at $1/g^2 = 1.0$ even an 8^3 lattice is too small for hadrons which are much lighter than the upsilon family.

It is hard to avoid the conclusion that the results obtained until now have not much relevance to the basic question concerning (quenched) QCD. On the other hand there are techniques[12] (the method we present here might be among them) which can be used on significantly larger lattices, opening the way towards reasonable calculations.

We defined the fermions on the lattice as suggested by Wilson.[13] The amplitude of moving a quark by one lattice unit is proportional to the hopping parameter K. The hopping-parameter expansion[5,13,14] is an expansion in terms of the length of quark paths. It is like a high-temperature expansion in statistical physics complicated by the presence of a background gauge field over which the quarks propagate.

Consider the quark propagator $D_{(0,a,\alpha)}(n, b, \beta; K)$, where $0, a, \alpha$ (n, b, β) are the initial (final) space, color, and spinor indices. Let us expand this propagator in terms of K,

$$D_{(0,a,\alpha)}(n, b, \beta) = \sum_{l=l_{\min}}^{\infty} \varphi_{(0,a,\alpha)}{}^{(l)}(n, b, \beta) K^l, \quad (1)$$

where $l_{\min}$ is the length of the shortest path between 0 and n. φ satisfies the following iteration equation:

$$\varphi_{(0,a,\alpha)}{}^{(l)}(n,b,\beta)=\sum_{\mu=1}^{4}\varphi_{(0,a,\alpha)}{}^{(l-1)}(n-\hat{\mu},c,\gamma)U_{n-\hat{\mu},\mu}{}^{cb}(1-\gamma_\mu)_{\gamma_\beta}$$
$$+\sum_{\mu=1}^{4}\varphi_{(0,a,\alpha)}{}^{(l-1)}(n+\hat{\mu},c,\gamma)U_{n,\mu}{}^{+cb}(1+\gamma_\mu)_{\gamma_\beta}. \tag{2}$$

By adding the initial condition $\varphi_{(0,a,\alpha)}{}^{(0)}(n,b,\beta)=\delta_{0n}\delta_{ab}\delta_{\alpha\beta}$, this equation can be used to calculate the expansion coefficients iteratively.

The gauge field configurations were generated by Monte Carlo techniques using Wilson's gauge field action. We have collected the results from 50 initial points on five independent gauge field configurations at $1/g^2=1.0$.

In 24th order the hadron can propagate 12 lattice distances along the axes; therefore a $(24)^4$ background gauge field configuration would be necessary in principle. This task was beyond our possibilities. We used 8^4 periodic gauge configurations which were *copied* over the lattice [$3^4=81$ times]. The quarks were not constrained by boundaries in their propagation (case I).

It is easy to show that a pole in the hadron propagator in momentum space implies a pole in the propagator in K.[5] Padé analysis was used to identify the expected poles of the Fourier-transformed propagator.

In order to get additional information on the finite-size effects and also to get better contact with other analyses,[1-4,6-9] an independent hopping-parameter expansion was performed in a finite 8^4 box, where periodic boundary conditions were used both for the gauge fields and for the fermions (case II). In this case the computer time increases only linearly with the order of the calculation. 32nd order series were obtained at $1/g^2=1.0$ using 32 starting points. The propagators were analyzed as in Refs. 1–4 and 6–9 by studying their exponential decay in configuration space.

We shall concentrate on the results obtained in the pseudoscalar and vector meson channels at $1/g^2=1.0$. Consider first the results from case I.

The meson propagator $D_{\text{meson}}(p_0=iMa,\vec{p}=\vec{0};K)$, which is obtained (in case I) as a 12th order series in K^2, is Padé analyzed in K^2. The expected poles in K should establish the relation between the meson mass M and the quark mass (related to K).

The Padé table is very stable for all Ma in the pion channel, and for $Ma \gtrsim 0.8$ in the rho channel. A closer look reveals, however, that something very strange is going on. Every pole is followed immediately by a zero. [For instance, at $(Ma)^2=1.2$, $K_1(\text{zero})-K_1(\text{pole})=0.0008\pm0.0001$, $K_2(\text{zero})-K_2(\text{pole})=0.008\pm0.003$]. By interpreting the first and the second poles as π and π', respectively, the presence of a nearby first zero would imply that the π' is coupled to this channel ~10 times stronger than the π itself—an unacceptable conclusion. If we would forget this immediate problem (we cannot), the value of K_c (where "m_π" $=0$) and the "masses" would be as "reasonable" as in the previous analyses (Fig. 1).

It is suggestive that actually a cut is simulated by the repeated pole-zero structure in the Padé approximants. Where could this cut come from?

As a result of the copied, periodic gauge field configurations the quarks move in a periodic potential. *If* the size of the (would be) bound state is larger than (or comparable to) the length period ($8a$ in our case), then strong tunneling is ex-

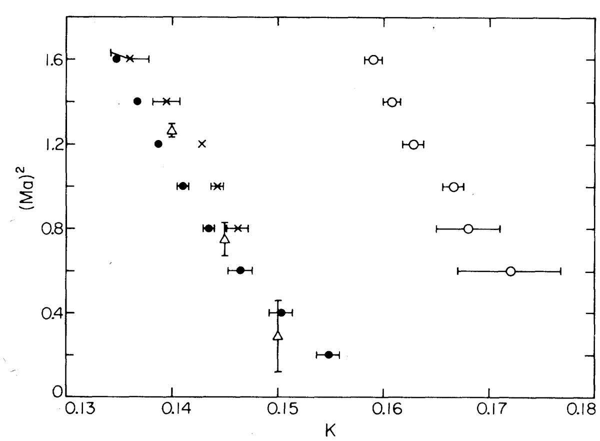

FIG. 1. The positions of the first and the second poles in the pseudoscalar channel, and that of the first pole in the vector channel, denoted by solid circles, open circles, and crosses, respectively. For $(Ma)^2 < 0.8$ the first pole-zero pair cannot be identified in the Padé table of the rho meson. The triangles give the lowest mass in the pion channel obtained in a periodic box (case II). The errors refer to uncertainty coming from the Padé analysis only.

pected and the resulting state will be a Bloch wave instead of a bound state. The discrete energy levels (poles in the propagator) will be dissolved into bands (cuts in the propagator). Strong tunneling implies that the bands overlap and the original structure is spoiled completely. If, however, the finite-size effects are under control, then the tunneling is weak, the bandwidth is small, and the narrow bands simulate poles for all practical purposes. Between these two extreme situations, the bandwidth is a quantitative indicator of the finite-size problems. Of course, in a finite periodic box (as in Refs. 1–4 and 6–9 and in case II) we always get poles, however large the finite-size effects are. One expects, however, that the distortion of the spectrum is of the same order as the bandwidth in the copied case.

For small masses Ma, the expansion coefficients of the meson propagator $D_{\text{meson}}(iMa, \vec{p}=\vec{0}; K)$ indicate a much weaker singularity than a pole. Actually the coefficients of the pion propagator at $Ma=0$ are consistent with a singularity like $\sim(c-K^2)^2\ln(c-K^2)$, which is the singularity of the free field theory.[13] By increasing the quark masses the singularity gets stronger (as expected), but painfully slowly. For instance, at $(Ma)^2=1.2$ the series is extremely well described by the function $(c^2-K^2)\ln(c^2-K^2)$ with $c=0.1309$ as is shown by Table I.

For heavier quarks (the charmonium and upsilon region) the size of the meson becomes smaller, the tunneling becomes weaker, and one expects well-separated bands with decreasing width as the quark mass is increased. In this region the Padé table contains two stable poles (and a zero between them) approaching each other as the quark mass is increased (Fig. 2). We shall see below that this behavior is consistent with a band picture obtained in a nonrelativistic potential model.

For heavy quarks nonrelativistic quantum mechanics should give a good qualitative description. To stay as close to the original lattice calculation as is possible, the Schrödinger equation was solved on a lattice (by the usual replacement of the Laplace operator). The potential is a sum of a Coulomb and a linear term. The parameters were fixed to give a good overall spectrum in the infinite-volume limit.

The results are summarized in Fig. 3. The bandwidths of the first two S states are given as a function of the quark mass for the situation when the potential is repeated periodically with a periodic length of 1 fm. The discrete levels of a periodic box of size of 1 fm are also shown. The resolution is $a=0.167$ fm, and we checked that the results are almost the same at $a=0.2$ fm.

The behavior of the width of the first band in Fig. 3 is qualitatively the same as that of the distance between the poles in Fig. 2. The levels of the periodic box are shifted by ~ 250 and ~ 200 MeV relative to the exact solution for the charmonium $1S$ and $2S$ states, respectively. These shifts have the same order of magnitude as the

TABLE I. The hopping-parameter series $[(C_{n-1}/C_n)^{1/2}]$ of the pion propagator at $(Ma)^2=1.2$ compared with that of the function $(c^2-K^2)\ln(c^2-K^2)$ with $c=0.1309$. C_n is the coefficient of K^{2n}.

n	Pion propagator at $(Ma)^2=1.2$	$(c^2-K^2)\ln(c^2-K^2)$ $c=0.1309$
12	0.1423	0.1423
11	0.1434	0.1434
10	0.1448	0.1447
9	0.1465	0.1464
8	0.1485	0.1484
7	0.1512	0.1512
6	0.1549	0.1549

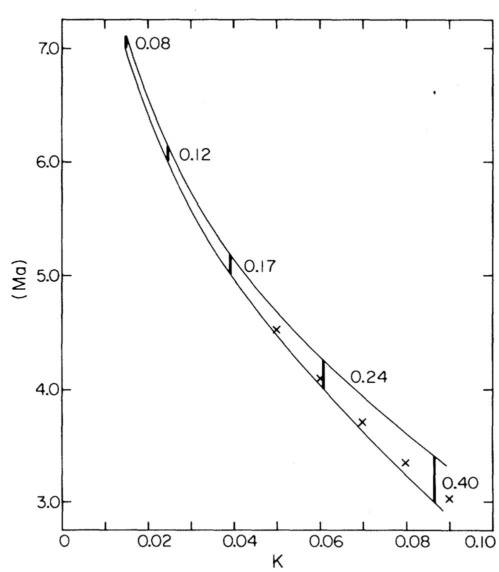

FIG. 2. The behavior of the first two poles in the pion channel as a function of K with copied gauge fields (case I). The distance between the poles is given in dimensionless units. The lowest-energy level obtained in a periodic box (case II) is denoted by crosses.

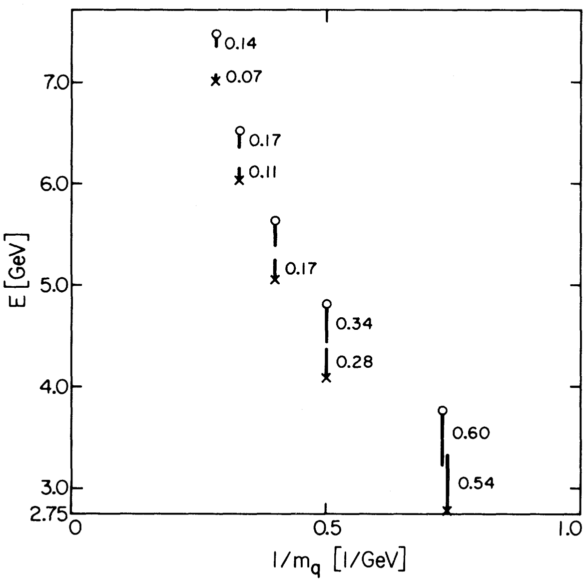

FIG. 3. The bandwidths of the $1S$ and $2S$ states as a function of the inverse quark mass as obtained by solving the Schrödinger equation on a lattice. The potential is periodic with a periodic length of 1 fm. Within a box it is given by $V(r) = -\alpha/r + \lambda r$, where $\alpha = 0.51$ and $\lambda = 0.17$ (GeV)^2. The resolution is $a = 0.167$ fm. The numbers give the width in gigaelectronvolts. The crosses and circles give the discrete $1S$ and $2S$ levels, respectively, in a single box with periodic boundary conditions.

widths themselves. In this situation it is very difficult to interpret the two poles in Fig. 2 as two well-separated bound states describing the $1S$ and $2S$ charmonium states.

Let us remark that the higher-order expansion coefficients obtained in the 8^4 periodic box (case II) differ significantly from the corresponding coefficients of the copied case (I). This fact is already a warning of the strong finite-size effects.

Finally we note that none of our conclusions would be changed by using the results obtained from ten initial points only on a single background configuration.

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