How to Get an Upper Bound on the Higgs Mass

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In view of the almost established triviality of φ^4 in four dimensions it is conjectured that the mass of the Higgs particle in the minimal model is bounded by an amount that can be estimated without the knowledge of the physics at shorter distances. A possible numerical experiment which could give a nonperturbative estimate for the bound is proposed.

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It is generally suspected that an interacting continuum φ^4 theory does not exist in four dimensions.[1] Specifically, there is no known sequence of well defined, regularized systems which admits a φ^4 theory in four dimensions with nonvanishing coupling in the limit of infinite cutoff. There are various ways to look for such a sequence. Although dimensional regularization, as presently understood, does not really define a cutoff model outside perturbation theory the renormalization-group equations seem to force the physical coupling into a narrow range when the bare coupling is varied over all physically acceptable values. In the limit of four dimensions this range collapses to a point and the physical coupling vanishes.[2] Another approach to the problem of establishing the existence of an interacting renormalized field theory is to look for a nontrivial fixed point in a class of short-range systems with the appropriate symmetries.[3] No such point has been found for φ^4. In the search a variety of methods were used: approximate renormalization-group recursion relations,[4] high-temperature expansions,[5] and direct numerical simulations.[6] All of these methods are outside weak-coupling perturbation theory. Moreover, several authors have rigorously proven theorems which appear to come close to showing the triviality of φ^4 in four dimensions.[7] The $1/N$ expansion when applied to a $(\sum_i \varphi_i^2)^2$ model, regularized in any of a number of ways, essentially gives the perturbative renormalization-group result: The physical coupling has to vanish if we demand the sequence to make sense for a finite cutoff and a finite N.[8]

From a pragmatic point of view there is little difference between a theory with a true (interacting) continuum limit and an effective theory with a cutoff at some very high, experimentally inaccessible mass. In a φ^4 theory it is believed that the ratio of the cutoff to the renormalized mass m_R is bounded from above and its maximum is a decreasing function of the renormalized coupling λ_R. For small λ_R, this ratio can be very large and as an effective theory φ^4 makes perfectly good sense, but as λ_R becomes large the cutoff required to maintain it moves down toward m_R. When λ_R increases to the point where the cutoff is comparable to m_R the model ceases to make any sense, even as an effective theory. The conclusion is that a φ^4 theory is physically meaningful only if the coupling is sufficiently weak.

In the Weinberg-Salam model the gauge and the Yukawa couplings are known to be small and can be treated perturbatively.[9,10] If the self-coupling of the Higgs field were to become large, it would therefore renormalize in essentially the same way as does λ_R in a pure φ^4 theory. But a strong-coupling φ^4 theory seems to be impossible[11] and it is likely that the Weinberg-Salam theory is physically meaningful only if the Higgs self-coupling is weak. The purpose of the present note is to point out that this implies an upper bound on the Higgs mass[12] which we estimate to be of order 1 TeV. This value is close to the estimate of Lee, Quigg, and Thacker[13] who showed that for a heavier Higgs particle the tree approximation must break down. Our results indicate that higher orders in perturbation theory cannot solve the problem. The bound is, in principle, not exactly computable without a more complete theory; however, because of perturbative renormalizability it presumably can be rather well approximated without any information beyond the standard model. Our estimate is, at best, rough; we will also describe a numerical experiment which, if carried out, would yield a precise number when the theory is regulated by a lattice. The closer to the bound the actual Higgs mass turns out to be the lower is the energy at which new physics has to come into play. Only with a relatively light Higgs particle is a "desert" up to grand unification scales theoretically possible.

The basic observation is very simple. The Higgs sector with one complex doublet Φ is defined by

$$L_{\rm H} = \tfrac{1}{2}(\partial_\mu \Phi)^\dagger(\partial^\mu \Phi) - \tfrac{1}{2}m_0^2 \Phi^\dagger\Phi - \tfrac{1}{4}\lambda_0(\Phi^\dagger\Phi)^2 \qquad (1)$$

and at the tree level we have

$$(m_{\rm H}/m_W)^2 = 8\lambda_0/g^2, \qquad (2)$$

where $m_{\rm H}$ and m_W denote the Higgs and W masses and g is the SU(2) gauge coupling. The mass ratio in (2) can be used as a definition of the physical coupling λ. Then, in order to increase $m_{\rm H}$ while m_W and g are kept fixed, we have to increase λ by decreasing the cutoff. Since it is meaningless to have a cutoff smaller than the Higgs mass the latter cannot get arbitrarily large.[14] To get a rough estimate we use the perturbative (or leading logarithm) result

$$1/\lambda \geqslant (3/2\pi^2)\ln(\Lambda/\mu) \qquad (3)$$

and we obtain from (2)

$$\frac{m_{\rm H}}{m_W} \leqslant \frac{4\pi}{g\sqrt{3}}\,\frac{1}{(\ln\Lambda/\mu)^{1/2}} \approx \frac{900\ \text{GeV}}{m_W}\,\frac{1}{(\ln\Lambda/\mu)^{1/2}}. \qquad (4)$$

Working toward a more precise description of the procedure to obtain the bound we first introduce some simplifications. The U(1) part and the Yukawa couplings are disregarded and the gauge coupling is assumed to be sufficiently small to validate keeping only leading terms in g. This allows us to define the mass ratio (2) entirely within the Higgs sector. Such a definition includes all orders in λ. In an O(4) [$\approx$SU(2) $\otimes$ SU(2)] notation the Higgs sector is identical to the Gell-Mann–Levi linear sigma model[15]:

$$\Phi \equiv \begin{pmatrix} \sigma + i\pi_3 \\ \pi_1 + i\pi_2 \end{pmatrix}, \quad L_{\rm H} = \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}(\partial_\mu\vec{\pi})^2 - \tfrac{1}{2}m_0^2(\sigma^2+\vec{\pi}^2) - \tfrac{1}{4}\lambda_0(\sigma^2+\vec{\pi}^2)^2. \qquad (5)$$

The breaking of SU(2) $\otimes$ SU(2) down to its diagonal subgroup produces the $\vec{\pi}$ Goldstone particles which give the W's a mass, $m_W^2 = \frac{1}{4}g^2 f_\pi^2$, via a pole in the W-vacuum polarization[16]:

$$\vec{J}_\mu^{\,\rm H} = \sigma\partial_\mu\vec{\pi} - \vec{\pi}\partial_\mu\sigma, \quad \langle 0|\,\vec{J}_\mu^{\,i\,\rm H}(x)\,|\pi^j(k)\rangle = if_\pi k_\mu\,\delta^{ij}e^{-ikx}. \qquad (6)$$

In our approximation f_π can be computed from the Higgs sector alone.

For a Monte Carlo simulation we propose to use the following latticized form of the Euclidean version of (5):

$$S = \textstyle\sum_x \{\sum_\mu \frac{1}{2}\rho(x+\mu)\rho(x)\,{\rm tr}[U(x+\mu)U^\dagger(x)] + \frac{1}{2}\mu_0^2\rho^2(x) + \frac{1}{4}\lambda_0\rho^4(x) - 3\ln\rho(x)\}, \qquad (7)$$

$$Z = \int_0^\infty \prod_x d\rho(x)\int_{\rm SU(2)}\prod_x dU(x)\,e^{-S},$$

where ρ is a singlet under SU(2) $\otimes$ SU(2) and U transforms as $U \to VUW^\dagger$. The $\ln\rho$ term comes from the integration measure. This parametrization might permit the replacement of the SU(2) group manifold in which the U's take values by a (hopefully) sufficiently large discrete subgroup.[17] As a result, the simulation time could be reduced to an amount not significantly larger than the amount needed for one component φ.[4]

In the following we shall outline the procedure to obtain the bound on the Higgs mass. First the vacuum is aligned by a small symmetry-breaking term

$$\Delta S = -h\textstyle\sum_x {\rm tr}\,U(x) \qquad (8)$$

which has to be picked large enough to suppress the vacuum averaging induced by the finiteness of the system, but still sufficiently small that the contributions due to spontaneous symmetry breaking are much more important than those due to explicit breaking.

A "target" value of f_π will be picked: f_π^{-1} will be expressed as a finite, reasonable, fraction of the size of the system (in lattice units). By a procedure to be explained later on μ_0^2 will be varied starting from some large negative value until the "target" value of f_π is obtained. Next the Higgs mass is measured. For a fixed size of the system the process will be repeated for several values of the bare coupling $0 \leqslant \lambda_0 \leqslant \infty$. The inverse lattice spacing (actually the system's size in lattice units) acts as an ultraviolet cutoff. From this experiment we obtain $m_{\rm H}/m_W$ as a function of λ_0. This function is obtained for various values of the cutoff until, eventually, the maximal value (as a function of λ_0) of the Higgs correlation length becomes less than one lattice spacing. This gives us the bound we are after.

We now turn to the question of how to measure the Higgs correlation length and the pion decay constant. For the Higgs mass we need a correlation function which is sufficiently smooth in the infrared. (Some nonanalyticity in the infrared must appear because the massive particle is unstable and decays into pions.[18]) Thus we are led to consider O(4)-invariant quantities. Since we also need a controllable behavior in the ultraviolet the most natural choice is the following two-point function:

$$G(x) = \langle \rho^2(x)\rho^2(0)\rangle - \langle \rho^2\rangle^2 = \int \frac{d^4k}{(2\pi)^4} D_0(k) e^{ikx}. \tag{9}$$

In the continuum limit $G(x)$ has an additional ultraviolet divergence at $x=0$. A finite result will be obtained if a logarithmically diverging constant is subtracted from $D_0(k)$.[19] We therefore define the Higgs correlation by

$$\xi_H^2 = \frac{\sum_x \vec{x}^2 G(x)}{\sum_{\vec{x}\neq 0} G(x)}. \tag{10}$$

A simple but crude way to obtain an approximate value for the Higgs mass is to quench the variables U.[20] Alternatively one can replace the $\mathrm{tr}[U(x+\mu)U^\dagger(x)]$ term in (7) by its vacuum expectation value. If one considers a nonlinear generalization of (7) in which U is in SU(N), these two possibilities presumably correspond to $N=1$ and $N=\infty$, respectively. For consistency the result of Eq. (10) should come out at a value bounded by these two extremes.

The measurement of f_π proceeds by looking at the power falloff of the U correlation and also at the U vacuum expectation value:

$$\langle \mathrm{tr}\, U(x)U^\dagger(0)\rangle - \mathrm{tr}(\langle U\rangle)^2 \underset{x\to\infty}{\rightarrow} \frac{3}{8\pi^2}\frac{|z|^2}{\vec{x}^2}. \tag{11}$$

In the continuum z is defined (in Hilbert space notation) by

$$\langle 0|U_{\alpha\beta}{}^\dagger(x)|\pi^i(k)\rangle = \tfrac{1}{2} z \tau_{\alpha\beta}{}^i e^{-ikx}. \tag{12}$$

With use of standard current-algebra techniques it can be shown that

$$|\langle \mathrm{tr}\, U\rangle| = |f_\pi z|. \tag{13}$$

Equations (13) and (11) show that f_π can be measured without dealing with (the complicated) lattice current-current correlations.

If no unexpected problems arise we would estimate the computer time necessary for the implementation of the above project to be the equivalent of a few thousand central-processing-unit hours on a VAX 780/11 with a floating point accelerator.

In this note, we have advanced the conjecture that the Higgs particle of the minimal standard model must, in principle, as a result of requirements of mathematical consistency, have a mass smaller than some bound which can be estimated without the knowledge of the physics at shorter distances. A scan of all the energies up to about 1 TeV should therefore either find the Higgs particle or observe new physics. Only if a relatively low-mass Higgs particle is found is a "desert" still a theoretically open possibility.

The estimate of the bound we gave is rather naive; a Monte Carlo simulation of the type described could give a better number but would require a large amount of computer time. It would be useful if the $1/N$ expansion could be refined to include the $1/(N+8)$ and $1/(N+2)$ factors naturally and allow for the systematic computation of corrections.[21] This might be possible in view of the recent applications of $1/N$ to quantum mechanics.[22] Alternatively, the employment of approximate recursion relations might give some estimates for the bound which are of a nonperturbative nature. The availability of an analytic approximation would be useful in studying other Higgs-induced phenomena, such as the appearance of non-Abelian magnetic monopoles.

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[1]Fundamentally, this is because the theory has an infrared rather than ultraviolet fixed point at zero coupling.

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[20]This stabilizes the Higgs particle and should make it easier to measure the ρ correlation length.

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