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Blackbody Radiation Law: Quantum or Classical Explanation?

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The long-time behavior of a system of two nonlinear oscillators interacting through a linear continuous string is investigated numerically. The string is treated exactly (no spatial lattice). It is found that such classical systems display many features normally associated with quantum mechanics, such as a phase space which breaks up into an infinite number of finite-size cells of ergodic motion.

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Nonlinear continuous media have been studied numerically only indirectly, on a lattice, starting with the celebrated work of Fermi, Pasta, and Ulam.[1] The results[2] have consistently failed to support the classic point of view regarding the equipartition of energy. Perhaps they have not stirred more excitement in the physics community for two reasons: (i) The systems analyzed may possess an infinite number of conservation laws, rendering them effectively linear; (ii) the results could be a lattice artifact.

Actually, in the case of the blackbody problem, the continuous medium is linear. The nonlinearity is introduced into the problem through the interaction of the field with the nonlinear oscillators in the walls of the cavity. In this paper I will investigate a one-dimensional version of this problem (two nonlinear oscillators interacting through a linear string—Fig. 1). It has the advantage of being partly soluble, so that no spatial lattice is needed. The equations of motion are

$$\ddot{z} - z'' = 0, \quad x \neq \pm 1, \tag{1}$$

$$m_l \ddot{z}(-1, t) = \mu z'(-1, t) + F_l(z(-1, t)), \tag{2}$$

$$m_r \ddot{z}(1, t) = -\mu z'(1, t) + F_r(z(1, t)). \tag{3}$$

In writing them I have chosen units such that the length of the string is 2 and the speed of sound 1. The most general solution of Eq. (1) is

$$z(x, t) = f(t + x) + g(t - x). \tag{4}$$

Substitution into Eqs. (3) and (4) yields the following system of ordinary differential equations with time lag for f and g:

$$m_r \ddot{f}(t) = -m_r \ddot{g}(t - 2) - \mu[\dot{f}(t) - \dot{g}(t - 2)] + F_r(f(t) + g(t - 2)), \tag{5}$$

$$m_l \ddot{g}(t) = -m_l \ddot{f}(t - 2) + \mu[\dot{f}(t - 2) - \dot{g}(t)] + F_l(f(t - 2) + g(t)). \tag{6}$$

Specification of $f(t)$, $g(t)$, $\dot{f}(t)$, and $\dot{g}(t)$ on the interval $[0, 2]$ uniquely determines $f(t)$ and $g(t)$ for all t. It is trivial to verify that as a consequence of the equations of motion, the total energy

$$H \equiv \mu \int_{-1}^{1} dx \frac{\dot{z}^2}{2} + \frac{z'^2}{2} + \frac{m_l \dot{z}(-1, t)^2}{2} + V_l(z(-1, t)) + \frac{m_r \dot{z}(1, t)^2}{2} + V_r(z(1, t)), \tag{7}$$

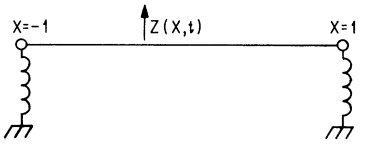

FIG. 1. Two nonlinear oscillators coupled through a linear continuous medium (string).

where

$$F_i(z) = -dV_i/dz, \tag{8}$$

is conserved. In the work presented here, I have

$$V_i(z) = k_i z^2/2 + \lambda_i z^4/4 + g_i |z|. \tag{9}$$

Equations (8) and (9) were integrated numerically, and the conservation of energy was used to verify the numerical accuracy. Before giving the details of the computation and the results, I would like to emphasize what outcome one would predict using the same line of thought as in deriving the Rayleigh-Jeans formula: The system, being nonlinear and (probably ?) sufficiently complicated, will wander with equal probability throughout its phase space of given total energy. The latter being infinite, the expectation value of, say, the kinetic energy of one of the particles will be zero. Moreover, since for the Gibbs distribution, ensemble averages and time averages,

$$\bar{A} = \lim_{T\to\infty} \bar{A}(T) = \lim_{T\to\infty} T^{-1}\int_0^T dt A(t), \tag{10}$$

are equal, one should observe that for any initial condition, the time-average kinetic energy of either particle tends to zero. Over my times of observation, this does not seem to be the case !

The numerical method was verified by applying it to a purely linear system ($\lambda_i = g_i = 0$, $i = 1, 2$), which can be analyzed analytically by normal-mode decomposition. Typically, the numerical accuracy is better and the time averages converge faster than in the case of nonlinear systems. I would like to remind the reader that in a linear system the equipartition of energy cannot occur, and this is verified numerically. Moreover, since the system is always performing some quasiperiodic motion, time averages depend (continuously) upon the initial conditions. This is shown in Fig. 2. The initial conditions were ($k_1 = k_2 = 1$, $m_1 = m_2 = 1$, $\mu = 1$)

$$f(x) = A\sin(\omega x + \pi/2), \quad g(x) = 0, \tag{11}$$

with A adjusted so that the total energy in the system was 10 in each run. The system was followed through 500 collisions and the accuracy was about

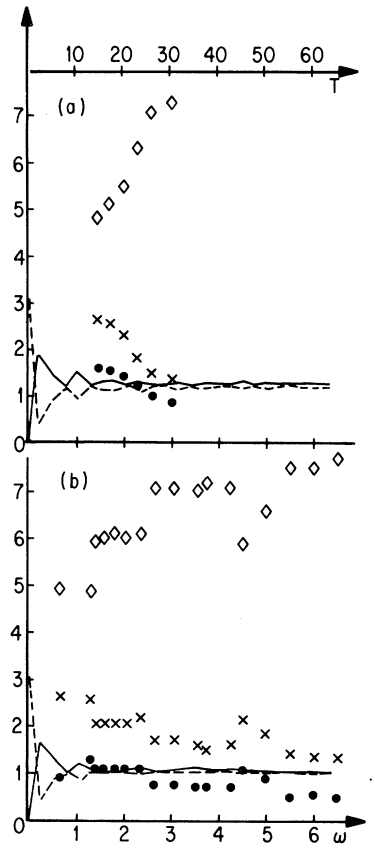

FIG. 2. Time-average values of the kinetic energy of the two balls vs time for $\omega = 2.35$ (continuous curves). The data points represent the time-average value of the kinetic energy (circles), particle energy (crosses), and string energy (lozenges) as a function of the initial conditions (ω) at constant total energy $E_T = 10$. (a) Linear system ($k_i = 1$). (b) Nonlinear system $g_i = 1$, $k_i = 0$.

10^{-3} per collision (one collision equals one interaction between the two balls).

As I have already stated, over my time of observation I detected no tendency for the time-average energies of the particles to go to zero. Under the assumption that the time of observation has been sufficiently long, this would rule out the Gibbs measure, leaving two possibilities: (i) the system is performing only quasiperiodic motions; (ii) the phase space is broken dynamically into an infinite number of finite-size ergodic cells.

As illustrated by the linear case [Fig. 2(a)], quasiperiodic motions yield time averages which change smoothly with the initial conditions. In Fig. 2(b) I show the dependence of the time-average kinetic energy, particle energy, and string energy with ω [see Eq. (11)] for the following non-

linear case:

$$k_1 = k_2 = \lambda_1 = \lambda_2 = 0, \quad g_1 = g_2 = 1 = m_1 = m_2, \quad E_T = 10. \tag{12}$$

Please notice that these time averages remain constant as ω varies over some nonzero length interval, then simultaneously change to a new value, and so on. Thus, dynamically the phase space breaks up into finite-size cells. The time-average value of any quantity takes only certain (discrete) values specified by the cell in which the starting point was chosen. I have verified that changing the nonlinear force to a purely cubic one ($g_i = k_i = 0$; $\lambda_i = 1$) leads to a similar behavior, but changes the actual cells (for instance in ω).

Within a given cell the system seems not to be mixing. Attempts to measure the Lyapunov exponent by studying the dependence of the correlation

$$G(\tau) \equiv \int_0^T dt\, z(t) z(t+\tau)$$

(z the position of one of the particles) with τ were unsuccessful: rather than falling exponentially with τ, G was oscillating. Thus the Lyapunov exponent must be very small if not zero. Another indication of weak or no mixing was obtained by starting the system at two nearby configurations in phase space and observing the two trajectories. For instance, the positions of the two particles were chosen as 0.2313 and -0.0243, and, respectively, 0.2213 and -0.0242. After 24 collisions they became -2.2268 and 0.3795, and, respectively, -2.0812 and 0.4354.

It is interesting to investigate also the distribution of the energy of the string among its normal modes. To this end let

$$z(x,t) = \sum_{n=0}^{\infty} a_n(t)\cos(n\pi x) + b_n(t)\sin[(n+\tfrac{1}{2})\pi x],$$
$$\dot{z}(x,t) = \sum_{n=0}^{\infty} c_n(t)\cos(n\pi x) + d_n(t)\sin[(n+\tfrac{1}{2})\pi x]. \tag{13}$$

The fraction of the string energy in the nth mode at time t is defined as (where E_S denotes string energy)

$$\alpha_n(t) = (1/E_S)(c_n^2 + n^2\pi^2 a_n^2),$$
$$\beta_n(t) = (1/E_S)[d_n^2 + (n+\tfrac{1}{2})^2\pi^2 b_n^2]. \tag{14}$$

In Fig. 3 I show the time-average value of α_n and β_n as a function of n for one of the runs specified in Eq. (12). This distribution seems to change with the initial conditions (ω) in the same manner as the other time averages: It is constant over some range; then it jumps to a new shape, etc. My data on the time-average values of α_n and β_n are poorer because of the large amount of computation time needed for Fourier analysis. However, in all the runs performed, the distribution of the energy of the string as a function of n is highly peaked and shows no tendency of becoming flat. Its actual shape depends upon the values of the parameters as well as upon the initial conditions. If all the parameters are maintained unchanged and the total energy increased, the peak broadens.

The final result I would like to report concerns a system with two different springs and masses:

$$k_i = 0, \quad m_1 = 1, \quad \lambda_1 = 1, \quad g_1 = 0, \quad m_2 = 2, \quad \lambda_2 = 0, \quad g_2 = 1. \tag{15}$$

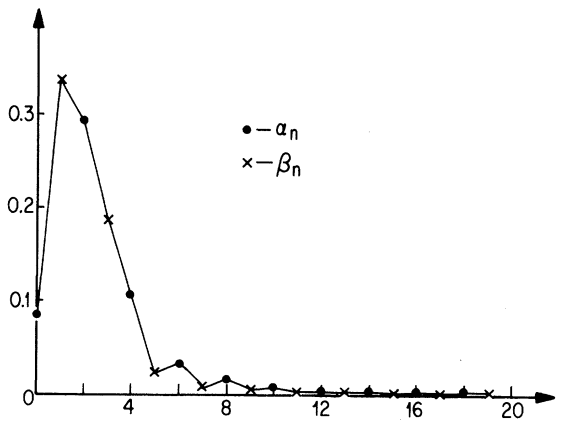

FIG. 3. Fraction of the string energy vs the number of nodes of the normal modes.

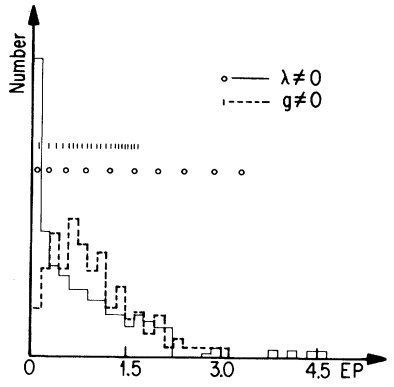

FIG. 4. Histograms of the energies of the two balls. For comparison, the quantum mechanical spectra are also shown. The $g|z|$ potential has a higher ground state, but a denser spectrum than the λz^4 one. Its histogram is also flatter and not peaked at zero energy.

In this case time averages converged at a slower rate. Consequently the system was followed for longer periods of time (up to 520 collisions). This led to a large violation of the total energy conservation (up to 90%). Thus these data are less reliable. It was verified by shorter runs (185 collisions), with higher accuracy performed on the Cyber. Qualitatively the results remained the same. Two features of the data for this system are noteworthy:

(i) In Fig. 4 I give the histogram of the total energy of each particle. They obviously depend upon the potential seen by the particle. For comparison I indicate also the corresponding quantum mechanical spectrum, with $\hbar$ determining the overall normalization, arbitrarily set equal to 0.125.

(ii) The time-average values of the kinetic energy of the two particles, when averaged over five runs having the same total energy, are very nearly equal (1.084 and, respectively, 1.054). The time-average values of the kinetic energies in individual runs ranged between 0.74 and 1.36.

The analysis presented here concerns classical continuous systems which have not been discretized. I believe it is their continuous nature which prevents such systems from reaching a regime in which the phase space consists of only one ergodic sea with islands of quasiperiodic motion. In the presence of a (spatial) lattice cutoff, at sufficiently high energy, such a regime would undoubtedly set in. For the continuous string, I have raised the total energy up to 2000 without qualitatively changing the results. The dependence of the time averages upon the initial conditions and the persistence of the behavior for different force laws rule out, in my opinion, the existence of an infinite number of conservation laws. Hence I would venture to guess that the behavior I observed is typical of nonlinear continuous media in one dimension.

Are these peculiarities of dynamics in one dimension? Or perhaps the times of observation were too short? I am in the process of answering these questions by new numerical experiments. For now I feel that the available numerical evidence lends credence to my hypothesis regarding a possible phenomenological nature of quantum mechanics.[3]

I would like to thank Willis Lamb for allowing me to use his computer, and Sandhya Devi for helping me with the technical aspects of programming.

Note added.—Since the submission of this paper, the accuracy and the speed of the program have been increased by 10^5 and 10^2, respectively. Thus it became possible to follow a system in three dimensions for up to 2400 interactions. Qualitatively the behavior of the system is similar to the one reported in this paper, although the presence of cells has not been established yet, because of the high cost of computing. These results will be reported shortly.

[1]E. Fermi, I. R. Pasta, and S. M. Ulam, Los Alamos National Laboratory Report No. LA-1940, 1955 (unpublished).

[2]For a review, see L. Galgani and A. Scotti, Riv. Nuovo Cimento 2, 185 (1972).

[3]A. Patrascioiu, "Cosmology and Quantum Mechanics" and "On the Nature of Quantum Behavior" (to be published, and *Beyond the Mystery of Quantum Mechanics*, Festschrift for F. E. Low (MIT Press, Cambridge, Mass., 1982).