Zero-Energy Theorem for Scale-Invariant Gravity

David G. Boulware
Physics Department, University of Washington, Seattle, Washington 98195

and

Gary T. Horowitz and Andrew Strominger
The Institute for Advanced Study, Princeton, New Jersey 08540

(Received 21 March 1983)

The general scale-invariant Lagrangian, $-\frac{1}{4}\int d^4x(-g)^{1/2}[\alpha c^{\mu\nu\lambda\sigma}c_{\mu\nu\lambda\sigma}+\beta R^2]$, with $\alpha\beta \geqslant 0$, is considered. Although this theory admits linearized solutions with negative energy, it is shown that all exact solutions representing isolated systems have precisely zero energy. This result also holds in the presence of arbitrary matter. It can be understood as resulting from a confinement of energy. The implications of this result for quantum gravity are discussed.

PACS numbers: 04.20.Cv, 04.20.Me, 04.60.+n

The conformally invariant Weyl action—the integral of the square of the Weyl tensor—is an attractive candidate for the fundamental action of quantum gravity. Quantum conformal gravity is probably renormalizable[1] and asymptotically free.[2] Since dynamical breakdown of scale invariance occurs generically in quantum field theory, we expect that, at long distances, the conformal invariance will be broken, leaving general coordinate invariance. The resultant theory will be described by an effective action of which the leading term at large distances is the Hilbert action,[3] with an in-principle calculable effective Newtonian constant.[4] Because of asymptotic freedom, conformal invariance is restored at short distances.

The principal reason that this theory has not received general acceptance is because, since it is a higher derivative theory, the solutions of the classical theory are expected to have negative energy and, therefore, exhibit instabilities. Indeed, there are solutions of the linearized equations with arbitrarily large negative (or positive) energies. In the quantum theory, this problem can be formally cured by quantizing with an indefinite metric in the Hilbert space.[5] The resultant theory is not unitary unless the perturbation expansion is modified by use of the Lee-Wick procedure.[5,6] However, this solution is contrived and its ultimate consistency is unclear.[7]

In this paper, we show that asymptotically flat solutions of the classical equations all have zero energy. The solutions to the linearized equations with nonzero energy of either sign do not correspond to the limit of a one-parameter family of exact solutions. As a result, one cannot draw conclusions from the linearized theory concerning the stability of the full quantum theory.

It should be emphasized that in no sense does the vanishing of the energy imply that the theory is trivial; the solutions include *all* solutions of the vacuum Einstein equations.

The physical basis of our result is simple: Because the action is fourth order, the (appropriately defined) classical potential grows linearly with distance. Another theory with a linear potential is *quantum* Yang-Mills theory. In that theory, we know[8] that only systems with zero total color have finite energy; i.e. color is confined. If the total color is not zero, there is necessarily a color field extending to infinity with divergent energy. In the case of scale-invariant gravity, only systems with zero total energy have finite energy. Thus we may say that energy is confined.[9]

If we break scale invariance by adding an Einstein term, the dynamics at large distances will be dominated by that term. The long-range potential will be Coulombic rather than linear, and hence no confinement is expected and the energy can probably have either sign.

The theorem we will prove applies to the general scale-invariant gravitational action,

$$w=-\tfrac{1}{4}\int d^4x\,(-g)^{1/2}\{\alpha c^{\mu\nu\lambda\sigma}c_{\mu\nu\lambda\sigma}+\beta R^2\}. \qquad (1)$$

The expression for the energy in this theory is different from that in general relativity. To see this, we briefly review the canonical formulation of scale-invariant gravity[9,10] which is based on the systematic treatment of higher derivative theories given by Ostrogradsky.[11]

Fix a spacelike surface, Σ, in a space-time, $(M, g_{\mu\nu})$, which is asymptotically flat (in a sense to be made precise shortly). The canonical variables are the three-metric, g_{ij}, a symmetric tensor density of weight one, Q^{ij}, and their conjugate momenta, p^{ij} and P_{ij}. The variables Q^{ij}, P_{ij} correspond to the extra degrees of freedom in higher-derivative gravity. The relation between Q^{ij}, P_{ij}, and the space-time curvature and extrinsic curvature, K_{ij}, of Σ is given by

$$Q^{ij} = g^{1/2}\{2\alpha C^{0i}{}_0{}^j + \beta g^{ij} R\}, \tag{2}$$

and

$$P_{ij} = 2K_{ij}. \tag{3}$$

Notice that, up to a factor, the trace-free part, Q^{Tij}, of Q^{ij} is the "electric" part of the Weyl tensor and that the trace of Q is just the scalar curvature. The formula for the momentum, p^{ij}, conjugate to g_{ij} involves the third time derivative of the metric and is more complicated.[10] Since its explicit form will not be required, we omit it.

Since this theory has a gauge invariance (the diffeomorphism group), there are constraints. As in general relativity, these are the time-time and time-space components of the classical field equations.[12] In terms of the canonical variables, these constraints are

$$C = \tfrac{1}{2}p^{ij}P_{ij} - \alpha g^{1/2} C^{0ijk}C^0{}_{ijk} - Q^{Tij}Q^T{}_{ij}/2\alpha g^{1/2} - Q^2/36\beta g^{1/2} - \tfrac{1}{4}Q^{ij}P_{ij}P - \tfrac{1}{8}Q(P^{ij}P_{ij} - P^2) + \tfrac{1}{2}\,{}^3RQ - {}^3R_{ij}Q^{ij} - D_iD_jQ^{ij} = 0, \tag{4}$$

and

$$C_k = \tfrac{1}{2}Q^{ij}D_kP_{ij} - D_i(P_{jk}Q^{ij}) + D_i p_k{}^i = 0, \tag{5}$$

where D_i, ${}^3R_{ij}$, and ${}^3R$ are the covariant derivative, Ricci tensor, and scalar curvature of g_{ij}; P is the trace of P_{ij}, and $C^0{}_{ijk}$ can be expressed in terms of the spatial derivatives of P_{ij}.[10]

The generators of diffeomorphisms are given by volume integrals of the constraints up to surface terms. These surface terms are fixed by the requirement that the generators be differentiable functions[13] on the phase space, $\Gamma = \{g, p, Q, P\}$. If the functions were not differentiable, then of course they would not generate any transformations. These generating functions are

$$H_N = \int N(x)C(x)d^3x + \int ND_jQ^{ij}\,dS_i \tag{6}$$

and

$$P_{Nk} = \int N^k(x)C_k(x)d^3x - \int N^k\{p_k{}^i - Q^{ij}P_{jk}\}\,dS_i, \tag{7}$$

where N, N^k are any (asymptotically well behaved) function and vector field on Σ. In addition, differentiability requires some weak falloff conditions on the canonical variables. These are not relevant to our proof and will not be presented. If N and N^k approach constant nonzero values asymptotically, then H_N and P_{Nk} generate asymptotic time and space translations. They are therefore identified with the Hamiltonian and momentum. The total energy E, is simply the value of the Hamiltonian when the constraints are satisfied (and N approaches 1 asymptotically),

$$E = \int D_jQ^{ij}dS_i. \tag{8}$$

Similarly, the momentum is given by

$$P_k = -\int\{p_k{}^i - Q^{ij}P_{jk}\}dS_i. \tag{9}$$

We can now make more precise the sense in which the space-time is asymptotically flat. We wish to consider only those solutions with finite energy and momentum. We therefore require[14] that

$$\begin{aligned} &g_{ij} = \delta_{ij} + O(r^{-1}), \quad p^{ij} = O(r^{-2}), \\ &Q^{ij} = O(r^{-1}), \quad P_{ij} = O(r^{-1}), \end{aligned} \tag{10}$$

and that the spatial derivatives of these fields fall off faster by one power of r^{-1}. This completes our review of the canonical formulation of the theory.

Notice that the expressions for energy and momentum in this theory are very different from those of general relativity. For example, the energy is given by the r^{-1} part of Q^{ij} but, by (2), this corresponds to the r^{-1} of the *curvature*. Recall that the Arnowitt-Deser-Misner energy for asymptotically flat solutions in general relativity is given by the r^{-1} part of the *metric*, and the curvature typically drops off like r^{-3}.

For what follows, it is convenient to reexpress the total energy, E, in terms of a volume integral over Σ. By Gauss's law and (8), we have

$$E = \int D_iD_jQ^{ij}d^3x. \tag{11}$$

With use of the constraint (4), this becomes

$$E = \int\big[\tfrac{1}{2}p^{ij}P_{ij} - \alpha g^{1/2}C^{0ijk}C^0{}_{ijk} - Q^{Tij}Q^T{}_{ij}/2\alpha g^{1/2} - Q^2/36\beta g^{1/2} + {}^3R_{ij}Q^{ij} - {}^3RQ/2 - \tfrac{1}{4}Q^{ij}P_{ij}P - \tfrac{1}{8}Q(P^{ij}P_{ij} - P^2)\big]. \tag{12}$$

Since the expression for energy in this theory is probably unfamiliar, we now briefly consider the linearized theory. The fields, $g_{ij}=\delta_{ij}$, $Q_{ij}=0$ $=p_{ij}=0=P_{ij}$, clearly satisfy the constraints and represent a plane in flat space-time. When we linearize about this solution, the constraint equations become

$$\partial_i \partial_j Q^{ij}=0 \tag{13}$$

and

$$\partial_i p^{ij}=0. \tag{14}$$

The first-order changes in g_{ij} and P_{ij} are not constrained. Let $Q^{ij}=0$ and p^{ij} be an arbitrary solution to (14) satisfying the boundary conditions (10). The energy for this solution evaluated to quadratic order is easily obtained from (12):

$$E=\int\{\tfrac{1}{2}p^{ij}P_{ij}-\alpha g^{1/2}C^{0ijk}C^{0}{}_{ijk}\}d^3x. \tag{15}$$

Recall that $C^0{}_{ijk}$ involves only spatial derivatives of P_{ij}. Since P_{ij} is arbitrary, we can choose $P_{ij}=\lambda p_{ij}$. If we choose λ sufficiently small, then the second term is negligible compared to the first; in that case,

$$E=\lambda\int p^{ij}p_{ij}\,d^3x, \tag{16}$$

which clearly can have either sign depending on the sign of λ. Thus, there exist linearized solutions to the constraint equations with either positive or negative energy.

We now state and prove our main theorem.

Theorem.—If (g,p,Q,P) satisfy the constraint (4) with $\alpha\beta\geqslant 0$, and the boundary conditions (10), then $E=0$.

The proof is remarkably simple. Consider Eq. (12). From the boundary conditions, (10), we see that all terms on the right-hand side fall off faster than r^{-2}, except for the two Q^2 term. Thus,

$$E=-\int\{Q^{Tij}Q^T{}_{ij}/2\alpha g^{1/2}+Q^2/36\beta g^{1/2}+h\}d^3x, \tag{17}$$

where h vanishes at least as fast as r^{-3}. Now, if $E\neq 0$, we see from (8) that Q^{ij} must have a nonzero r^{-1} contribution, and hence the integral of $Q^{ij}Q_{ij}$ diverges. Since α and β have the same sign by assumption, the first two terms in (17) cannot cancel. Since the boundary conditions require E to be finite, we obtain a contradiction; hence, $E=0$. Q.E.D.

It follows immediately from this theorem that the linearized solutions with nonzero energy cannot correspond to a one-parameter family of exact solutions. In fact, one cannot even extend these linearized solutions to second order. The second-order change in Q^{ij} will have an r^{-1} term which will cause the energy to diverge.

The above theorem is the analog of the positive-energy theorem in general relativity.[15] As in that case, the theorem remains valid in the presence of matter. However, since the proof only requires that the constraint equation hold asymptotically, it is sufficient here that the energy density of the matter fields fall off faster than r^{-2}; one does not need to impose the dominant energy condition.

We now discuss the implications of our results for quantum gravity. The hope is that this classical zero-energy theorem has a quantum analog that will cure the instabilities in quantum perturbation theory. The full quantum theory will be governed by an effective action. At long distances, the effective energy will be the Arnowitt-Deser-Misner energy, not that defined in (11). Therefore, in order to investigate the question of instabilities, one must understand the transition from short- to long-distance behavior. This is, of course, a difficult dynamical question. What we have shown here is that previous analyses indicating instabilities of the quantum theory were incorrect because they assumed linearization stability. This issue therefore deserves a thorough reinvestigation.

Since the action is positive definite for $\alpha,\beta\geqslant 0$, it may be possible to define the theory by means of a Euclidean formulation. In constrast to the real-time, Minkowskian, formulation with zero energy for all solutions, the Euclidean functional integral formulation may be simple because there is only one zero-action solution: flat R^4. However, the functional integral does have other asymptotically Euclidean extrema.[16] These instantons are a close analog of the familiar Yang-Mills instantons. Presumably, an adaptation of the techniques developed for the Yang-Mills case can be used here. The instantons may be relevant to the problem of describing the breakdown of scale invariance.[17]

Finally, we comment on the connection between this result and the existence of conformal supergravity.[18] The fact that the Hamiltonian is formally the square of the supercharge suggests that the energy should be nonnegative[19]; that result, plus positivity of the Hilbert space metric, implies that positive-energy states must exist unless the supercharge vanishes identically.[20]

We are grateful to D. C. Christodoulou and E. Tomboulis for discussions. This work was supported in part by the U. S. Department of Energy under Contracts No. DE-AC06-81ER40048 and No. DE-AC02-76ER02220. One of us (G. T. H.) acknowledges receipt of an Albert Einstein fellowship.

---

[1]The renormalizability of a general fourth-order action has been shown by K. S. Stelle, Phys. Rev. D 16, 953 (1977). The question here is whether the Weyl action, the most general conformally invariant fourth-order action, requires the presence of nonconformally invariant counterterms in the bare action. A. Zee, Phys. Lett. 109B, 183 (1982), has shown, for Weyl gravity coupled to a Yang-Mills field, that no nonconformally invariant counterterms are required to all orders in the Yang-Mills coupling. Since the essential ingredient of his argument is asymptotic freedom, it may well hold to all orders in the gravitational coupling as well.

[2]S. Fradkin and A. A. Tsetlyin, Nucl. Phys. B201, 469 (1982); J. Julve and M. Tonin, Nuovo Cimento B64, 33 (1981).

[3]S. L. Adler, Rev. Mod. Phys. 54, 729 (1982), where a complete set of references is given.

[4]Adler, Ref. 3; A. Zee, Phys. Rev. D 23, 858 (1981); N. N. Khuri, to be published.

[5]T. D. Lee and G. C. Wick, Phys. Rev. D 2, 1033 (1970). See also A. Strominger, in *Ahead of his time: Bryce S. DeWitt, a collection of essays in honor of his 60th birthday*, edited by S. M. Christensen (Hilger, Bristol, 1983); E. Tomboulis, *ibid.*

[6]R. E. Cutkosky, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, Nucl. Phys. B12, 281 (1969).

[7]D. G. Boulware and D. Gross, to be published.

[8]E. Tomboulis, to be published.

[9]This is consistent with the ideas discussed by M. Kaku, Nucl. Phys. B203, 285 (1982), and Phys. Rev. D, to be published (two Articles).

[10]D. G. Boulware, in *Ahead of his time: Bryce S. DeWitt, a collection of essays in honor of his 60th birthday*, edited by S. M. Christensen (Hilger, Bristol, 1983). We use the notation of this paper, except for a factor of -2 in the definition of β. See also S. Kawasaki, T. Kimura, and K. Kitago, Prog. Theor. Phys. 66, 2085 (1981).

[11]M. Ostrogradsky, Mem. Acad. St. Pet. 6, 385 (1850). See also E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Dover, New York, 1944), p. 264.

[12]If $\beta = 0$, there exist additional constraints related to conformal invariance, but these play no role in our discussion.

[13]For a discussion of this in the context of general relativity, see A. Ashtekar and G. Horowitz, Phys. Rev. D 26, 3342 (1982); T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) 88, 286 (1974).

[14]Although these boundary conditions are probably the most natural choice to ensure finite energy and momentum, they are not, *a priori*, the weakest possible conditions. It is possible that the proof can be extended to include more general boundary conditions.

[15]R. Schoen and S. T. Yau, Commun. Math. Phys. 79, 381 (1981); E. Witten, Commun. Math. Phys. 80, 209 (1981).

[16]G. Horowitz and A. Strominger, to be published.

[17]B. Hasslacher and E. Mottola, Phys. Lett. 95B, 237 (1980).

[18]P. Van Nieuwenhuizen, Phys. Rep. 68, 191 (1981).

[19]S. Deser and C. Teitelboim, Phys. Rev. Lett. 39, 249 (1977); M. Grisaru, Phys. Lett. 73B, 249 (1978); G. Horowitz and A. Strominger, Phys. Rev. D (to be published); S. Deser, Phys. Rev. D (to be published).

[20]Further investigations on this question are currently in progress by D. G. Boulware, S. Deser, G. Gibbons, and K. S. Stelle.