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Configuration Spaces for Quantum Spinning Particles

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The authors investigate possible candidates of configuration spaces for the description of the rotational degrees of freedom of quantum spinning particles (including half-integer spin states), and point out their connections in relation to the set of physical observables and superselecting sectors present in the theory.

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Extended objects, in the form of strings, solitons, kinks, monopoles, etc.,[1] seem to play an ever increasing role in quantum field theory and elementary-particle physics. The very existence of extended objects makes it conceivable, on physical grounds, that their state space can be organized according to a Regge-like trajectory structure, where states of increasing value for the angular momentum are present, possibly with a highly unstable nature.

On the basis of this motivation we have analyzed the very simple case of the nonrelativistic rigid body, with the purpose of isolating some possible candidates of configuration spaces for the description of the rotational degrees of freedom in its quantum behavior, including half-integer spin states, and in relation to the assumed algebra of physical observables.

Our analysis is a byproduct of the program[2,3] of describing quantum spin in the framework of the approach to quantization provided by stochastic mechanics.[4]

In the classical case a nonrelativistic rigid body has configuration space given by SO(3). We assume that we have separated the center-of-mass motion described in R^3. We call $g \in \mathrm{SO}(3)$ the configuration of the system.

We assume a Lagrangian of the type[5]

$$L(g, \dot{g}) = \tfrac{1}{2} I \langle g^{-1}\dot{g}, g^{-1}\dot{g} \rangle , \tag{1}$$

where I is a constant having the dimension of inertial momentum, $g^{-1}\dot{g} \in \mathrm{su}(2)$, the Lie algebra of SO(3), and $\langle \cdots, \cdots \rangle$ is a suitable positive definite scalar product on su(2). For the sake of simplicity we consider only the case of a generic cylindric inertial ellipsoid. Since su(2) is isomorphic to R^3, by assuming a symmetry around the third axis, we can write, for $t = (t_1, t_2, t_3) \in \mathrm{su}(2)$,

$$\langle t, t \rangle = t_1^2 + t_2^2 + \epsilon t_3^2 , \tag{2}$$

with $0 \leq \epsilon \leq 2$. The value $\epsilon = 1$ corresponds to full symmetry (the spherical case), while $\epsilon = 0$ is the degenerate dipole case.

Upon quantization the right choice for configuration space is S_3 (the three-dimensional surface of the unit sphere in R^4) as the manifold of SU(2), the universal covering group of the classical configuration space SO(3). This is particularly clear when quantization methods are employed where path integrals play a central role, either in the Feynman version[6] or in the stochastic mechanics version.[2] Alternatively, according to the picture described by Misner, Thorne, and Wheeler,[7] we may assume that the SO(3) rigid body is connected to infinity with a fibration of topological strings and add to the configuration space the information about the topological entanglement of the strings. This procedure still extends to SU(2) the effective configuration space (see, for example, Guerra and Marra[8]).

Then the Hilbert state space is $\mathcal{H} = L^2(S_3, d\mu)$, where $d\mu$ is the symmetric normalized measure on S_3, corresponding to the invariant Haar measure on SU(2). On $\mathcal{H}$ there exist two representations of SU(2), coming from left and right multiplications, defined by

$$\begin{cases} \mathrm{SU}(2) \ni h \to U(h), & [U(h)\psi](g) = \psi(h^+ g), \\ \mathrm{SU}(2) \ni h \to V(h), & [V(h)\psi](g) = \psi(g h^+), \end{cases} \tag{3}$$

We call J_1, J_2, J_3 the generators of $U(h)$ and $\hat{J}_1, \hat{J}_2, \hat{J}_3$ those of $V(h)$. They satisfy the angular momentum commutation relations

$$\begin{aligned} &[J_1, J_2] = iJ_3, \quad [\hat{J}_1, \hat{J}_2] = -i\hat{J}_3 \quad \text{(cyclic)}, \\ &[J_i, \hat{J}_j] = 0. \end{aligned} \tag{4}$$

A very convenient parametrization of SU(2) is derived from its quotient with respect to the subgroup U(1). Then the corresponding homogeneous space is S_2, the two-dimensional spherical surface in R^3. Assuming spherical coordinates on S_2 given by (θ, φ), $0 \leq \theta \leq \pi$, $\varphi \in \mathrm{U}(1)$, and $\alpha \in \mathrm{U}(1)$, we can write the explicit expressions for the angular momentum operators:

$$\begin{aligned} &\hat{J}_3 = (\hbar/2i)\partial_\alpha, \\ &J_3 = (\hbar/i)\partial_\varphi + (\hbar/2i)\partial_\alpha = (\hbar/i)\partial_\varphi + \hat{J}_3, \\ &J^2 = -\hbar^2 \Delta_{S_3}, \end{aligned} \tag{5}$$

where Δ_{S_3} is the Laplace-Beltrami operator on S_3 given by

$$\Delta_{S_3}\psi = \Delta_{S_2}\psi + (1+\cos\theta)^{-1}(\partial_{\alpha\varphi}^{\;2}\psi + \tfrac{1}{2}\partial_\alpha^{\;2}\psi) \tag{6}$$

and Δ_{S_2} is the Laplace-Beltrami operator on S_2,

$$\Delta_{S_2}\psi = \sin^{-1}\theta\,\partial_\theta(\sin\theta\,\partial_\theta\psi) + \sin^{-2}\theta\,\partial_\varphi^{\;2}\psi. \tag{7}$$

Since $\alpha \in \mathrm{U}(1)$ the factor 2 in the expression of $\hat{J}_3$ assures the existence of half-integer eigenvalues for $\hat{J}_3$. Moreover, $J^2, J_3, \hat{J}_3$ can be considered as a complete set of compatible observables. They can assume the values $\hbar^2 j(j+1)$, $\hbar m$, $\hbar m'$, with $j = 0, \frac{1}{2}, 1, \ldots$; $m = -j, -j+1, \ldots, j$; $m' = -j, -j+1, \ldots, j$. Correspondingly each $\psi \in \mathcal{H}$ can be decomposed in the Peter-Weyl expansion

$$\psi(g) = \sum_{j,m,m'} R_{mm'}^{(j)*}(g)\psi_{mm'}^{(j)}, \tag{8}$$

where $R_{mm'}^{(j)}$ form a complete orthonormal system on S_3. Under left and right transformations (3) the components $\psi_{mm'}^{(j)}$ transform independently according to the given irreducible representation

$$\begin{aligned} &[U(h)\psi]_{mm'}^{(j)} = \textstyle\sum_{m''} R_{mm''}^{(j)}(h)\psi_{m''m'}^{(j)}, \\ &[V(h)\psi]_{mm'}^{(j)} = \textstyle\sum_{m''} \psi_{mm''}^{(j)} R_{m''m'}^{(j)}(h). \end{aligned} \tag{9}$$

The Hamiltonian can be written as

$$H = \frac{\hat{J}_1^{\,2} + \hat{J}_2^{\,2}}{2I} + \frac{\hat{J}_3^{\,2}}{2\epsilon I} = \frac{J^2}{2I} + \frac{(\epsilon^{-1}-1)\hat{J}_3^{\,2}}{2I}, \tag{10}$$

where $J^2, \hat{J}_3^{\,2}$ are explicitly given by (5).

The energy eigenvalues are

$$\begin{aligned} &E(j, m') \\ &\quad = \hbar^2 j(j+1)/2I + \hbar^2(\epsilon^{-1}-1)m'^2/2I; \end{aligned} \tag{11}$$

the corresponding eigenstates are found in the Peter-Weyl expansion (8). They are degenerate with respect to external rotations generated by the J's (as is obvious on physical grounds because there is no external magnetic field), and also with respect to internal rotations generated by $\hat{J}_3$. If $\epsilon = 1$ then we have complete degeneracy also with respect to internal rotations.

From (8) we see that the model contains in general coherent superpositions of all possible different spin values. This remark could play some role in the study of supersymmetric theories.

From (10) and (11) we see that, beyond the $(2j+1)$-fold degeneracy given by external rotations, there is an additional $(2j+1)$-fold degeneracy due to internal rotations in the symmetric case ($\epsilon = 1$), or an additional twofold degeneracy, connected with the $m' \to -m'$ symmetry, in the generic cylindric case, when $m' \neq 0$. In particular there will be in general four different orthogonal states, with the same energy, for the spin-$\frac{1}{2}$ subspace, corresponding to $m = \pm\frac{1}{2}$, $m' = \pm\frac{1}{2}$.

It is interesting to see what kind of mechanism can be provided, in the framework of this model, in order to remove unwanted additional degeneracy and lay down a bridge toward the world of elementary-particle physics. We consider here in particular two mechanisms: the reduction of the observable algebra and the limit to the dipole case $\epsilon \to 0$. In order to describe the first let us assume $\epsilon=1$ and let us remark that the physical content of the theory is not given by the Hilbert space of states but more precisely by the states on the algebra of observables.

Firstly let us consider the maximally coherent case. Let $\mathcal{A}$ be the algebra of all bounded observables on $\mathcal{H}$. Then each $\psi \in \mathcal{H}$, with $\|\psi\|=1$, will generate a state ω on $\mathcal{A}$ defined by

$$\omega(A)=\langle\psi, A\psi\rangle, \tag{12}$$

for each $A \in \mathcal{A}$.

As is very well known the state ω is invariant under the U(1) gauge group of rephasing $\mathcal{H}$, $\psi \to \exp(i\alpha)\psi$, $\alpha \in \mathrm{U}(1)$. In fact only the rays in $\mathcal{H}$ are physically relevant. The states ω are pure on $\mathcal{A}$, i.e., no nontrivial decomposition exists of the type

$$\omega(A)=\sum_i p_i\,\omega_i(A), \quad 0 \leq p_i \leq 1, \quad \sum_i p_i = 1. \tag{13}$$

But assume that we are interested in a smaller algebra of observables $\mathcal{A}$ and that we consider the states induced by $\mathcal{H}$ on $\mathcal{A}$ through (12). For example let $\mathcal{A}$ be the algebra of bounded observables invariant under any right transformation $V(h)$. Then it can be easily shown that $\mathcal{A}$ is generated by the $U(h)$'s, i.e., any element $\mathcal{A}$ can be approximated through linear combinations of the type

$$A=\sum_i \lambda_i\, U(h_i). \tag{14}$$

Then the states ω induced by $\mathcal{H}$ on $\mathcal{A}$ are defined by limits of the form

$$\omega(A)=\langle\psi, A\psi\rangle=\sum_i \lambda_i \langle\psi, U(h_i)\psi\rangle. \tag{15}$$

It is enough to consider

$$\omega(U(h))=\langle\psi, U(h)\psi\rangle=\int \psi^*(g)\psi(h^+g)\,d\mu(g). \tag{16}$$

Then by exploiting (8) we have

$$\omega(U(h))=\sum_j \mathrm{Tr}[\psi^{(j)+}R^{(j)}(h)\psi^{(j)}], \tag{17}$$

where $\psi^{(j)} \equiv \{\psi_{mm'}{}^{(j)}\}$ and the trace is taken in the $(2j+1)$-dimensional complex space.

Formula (17) shows that each state on $\mathcal{A}$ is a mixture of states corresponding to each different j. We see that in this case the gauge group $\mathcal{H}$ extends from the rephasing group to a gigantic U(1) $\otimes$ U(2) $\otimes \cdots \otimes$ U($2j+1$) $\otimes \cdots$, i.e., the state defined by ψ on $\mathcal{A}$ is identical to that given by ψ' if

$$\psi_{mm'}{}'^{(j)}=\sum_{m''}\psi_{mm''}{}^{(j)}A_{m''m'}{}^{(j)},$$

for any $A^{(j)} \in \mathrm{U}(2j+1)$. Moreover the restriction to $\mathcal{A}$ gives mixtures of Pauli-type representations.

In this case no nontrivial superpositions can be made for different spin states. Each spin state is described in the frame of a Pauli-like theory. The single irreducible representations become incoherent.

Since for each spin state j any complete set of observables is equivalent to a single operator with spectrum $-j, -j+1, \ldots, j$, then we can assume the configuration space in this case as given by T_{2j+1} (the discrete set with $2j+1$ objects). This picture leads to general path-integral representations written in terms of Poisson jump processes on T_{2j+1}; see for example Refs. 3 and 8. Both J^2 and $\hat{J}_3$ become superselected quantum numbers; the only surviving observable in the complete set is J_3 and we may form only coherent superpositions of its eigenstates.

An intermediate possibility is offered by the choice of $\mathcal{A}$ as the algebra of observables invariant under internal rotations around one axis (for example the third); then $\hat{J}_3$ becomes a superselected quantity with eigenvalue $\hbar m'$. States with different values of m' cannot be coherently superposed. In each m' sector J^2, J_3 may take the values $j(j+1)$, $j=|m'|, |m'|+1, \ldots,$ $m=-j, -j+1, \ldots, j$.

We see that each sector is organized in a hierarchy of states with different j, starting from m' and increasing by integer values. These systems represent trajectories with definite lower total angular momentum m' and only superposition of integer values (if m' is integer) on half-integer values (if m' is half-integer) but not both. This situation has some similarity with the phenomenological classification of particles in terms of Regge trajectories.

In this case, for each m' sector, the configuration space reduces to S_2. In fact the wave function ψ on S_3 has a trivial dependence on $\alpha \in \mathrm{U}(1)$ in the form

$$\psi=\hat{\psi}(\theta,\varphi)\exp(-2i\alpha m'). \tag{18}$$

The Hamiltonian on $L^2(S_2)$ is

$$H=-(\hbar^2/2I)[\Delta_{S_2}+2(1+\cos\theta)^{-1}(-im'\partial_\varphi - m'^2)], \tag{19}$$

as can be easily found from (6), taking into account that $\partial_\alpha \equiv -2im'$. Notice that H depends explicitly on m' if $m' \neq 0$.

The reader may be surprised by the fact that $L^2(S_2)$ can also accommodate states of half-integer spin values. In fact since $\varphi \in \mathrm{U}(1)$ the operator $-i\partial_\varphi$ can take only integer values, but it should be realized that the true angular momentum in this case is

$$J_3=(\hbar/i)\partial_\varphi - m', \tag{20}$$

as follows from (5). Therefore half-integer values are allowed if m' is such. For example for $m'=\frac{1}{2}$ the spin-$\frac{1}{2}$ wave function on $L^2(S_2)$ corresponding to $J_3=\hbar/2$ is

$$\hat{\psi}=[(1+\cos\theta)/2]^{1/2}. \tag{21}$$

In conclusion we can see that the dipole configuration space S_2 can also describe half-integer spin states provided the Hamiltonian (19) is chosen. Notice that in general the term $-im'\partial_\varphi$ acts like a kind of external magnetic field directed along the third axis, but its origin is purely kinematical and derives from the elimination of the U(1) degree of freedom attached to $\alpha \in$ U(1).

Finally let us consider the other degeneracy-removing mechanism based on taking the dipole limit $\epsilon \to 0$ in (10) and (11). We see that if $m' \neq 0$ all energy eigenvalues go to infinity depending on the value of m'^2. From a physical point of view only energy differences have meaning. Therefore in each subspace corresponding to a given value of m'^2 we can perform a different energy renormalization based on a simple shift and the eigenvalues become simply

$$E(j,m')=[\hbar^2 j(j+1)]/2I+E_{m'^2}, \tag{22}$$

where $E_{m'^2}$ is a residual renormalization constant. But now, in the limit $\epsilon \to 0$, no coherent superpositions of states with different m'^2 are possible, because the different renormalizations make phase differences between the different m'^2 sectors completely meaningless.

We see that the limiting theory $\epsilon \to 0$ splits into incoherent sectors defined by the values of m'^2. Therefore $\hat{J}_3{}^2$ becomes a superselected operator. On the other hand states with opposite values of m' can be superposed (because they have a common infinite renormalization). As a result a complete set of observables in each sector is now reduced to J^2, J_3, σ, where σ is the sign of $\hat{J}_3$, and we have $\hat{J}_3=\hbar\sigma|m'|$, $\sigma \in Z_2=\{-1,1\}$. In each sector we have now hierarchies of states with total angular momentum $j=|m'|$, $|m'|+1, \ldots$.

We see that the operator ∂_α assumes the form $\partial_\alpha=-i\sigma|m'|$. Therefore we can assume $S_2\times Z_2$ as configuration space; now the Hamiltonian is

$$H=-(\hbar^2/2I)[\Delta_{S_2}+2(1+\cos\theta)^{-1}(-i|m'|\sigma\partial_\varphi-m'^2)]+E_{m'^2}. \tag{23}$$

For $m'\neq 0$ a degeneracy connected to $m'\to -m'$ is still present.

In conclusion, starting from the maximally coherent configuration space $S_3 \equiv$ SU(2), we can arrive at S_2 if the allowed algebra of observables is invariant under any internal rotation around the third axis, or to the configuration space T_{2j+1} if the observable algebra is invariant under any internal rotation. In the dipole case $\epsilon \to 0$ the configuration space S_3 reduces to $S_2 \times Z_2$. Algebra restrictions or dipole limits produce incoherent mixtures of states according to the previous mentioned coherence breaking pattern.

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