# Relativistic Oscillator: Linearly Rising Trajectories and Structure Functions

A. O. Barut
*Department of Physics, The University of Colorado, Boulder, Colorado 80309*

and

H. Beker
*Department of Physics, Bogazici University, Bebek, Istanbul, Turkey*
(Received 7 March 1983)

A new class of realizations of the dynamical group O(4, 2) depending on an arbitrary function $G$ is found. As a special case it is applied to a moving relativistic composite object which in its rest frame is a three-dimensional oscillator. The wave equation is exactly solved; the generators of the Poincaré group are identified. It can be applied to a *moving* "bag" or bound state where the constituents are bound by a confining potential. The form factors and structure functions are discussed.

PACS numbers: 11.30.Na, 03.65.Ge, 11.10.Qr, 12.35.Kw

Various relativistic oscillatorlike systems have been considered to account for linearly rising trajectories of composite hadrons. The equation of Feynman, Kislinger, and Ravndal[1] and others makes use of four-dimensional potentials which involve a relative time: $V=(x_1{}^\mu - x_2{}^\mu)^2$. Hughston[2] considers two coupled Klein-Gordon particles with a common potential and arrives at a trajectory which for massless constituents is $M^2=4an+b$, $n=0,1,2,\ldots$. Rising mass trajectories have also been obtained from other algebraic models.[3] To make an oscillator-type system move relativistically is a long-standing unsolved problem. We derive here and solve exactly a covariant wave equation for a moving composite system which in the rest frame of the system reduces precisely to the three-dimensional oscillator problem, and obtain the trajectory formula which in the simplest case is *linear* in mass:

$$M_n = an+b = a(2s+l)+b, \quad s,l=0,1,2,\ldots.$$

We give explicitly the complete generators of the Poincaré group for this problem, the states and, as mentioned, the spectrum. The covariant wave equation immediately allows us also to give the form factors and structure functions if the system is coupled minimally to an external electromagnetic field. It can further be generalized to constituents with spin.

The theory is based on a class of new realizations of the dynamical group O(4, 2) characterized by an arbitrary function $G(r)$, which for special choices of $G(r)$ describe, in a unified manner, the quantum mechanical Kepler, oscillator, and Morse problems. Previously, the oscillator has been described by the symmetry group SU(3) and dynamical group SU(3, 1), and, for the Morse problem, only the radial group SO(2, 1) was known. The group SO(4, 2) allows us to identify the Lorentz subgroup SO(3, 1) and write a covariant relativistic wave equation for moving composite systems bound by harmonic forces in the same way as had been done earlier for the moving H atom.[4] It is not easy to identify the Lorentz subgroup in SU(3, 1).

The fifteen generators of the dynamical group O(4, 2) in terms of the coordinates $\vec{r}$ and $\vec{p}$ of the *relative motion* are given by

$$\begin{aligned}
&L_{ij}:\ \vec{L}_0=\vec{r}\times\vec{p};\\
&L_{i4}:\ \vec{A}=\left(\frac{1}{G'}-\frac{G}{2rG'^2}\right)\vec{r}K^2+i\left(\frac{1}{G}-\frac{1}{rG'}-\frac{G''}{2G'^2}\right)\vec{r}K+i\left(\frac{1}{G'}-\frac{r}{G}+\frac{rG''}{2G'^2}\right)\vec{p}+\left(-\frac{r}{G'}\right)K\vec{p}\\
&\qquad\qquad+\left(\frac{L_0{}^2}{2Gr}+\frac{GG'''}{4rG'^3}-\frac{3GG''^2}{8rG'^4}\right)\vec{r}-\left(\frac{G}{2r}\right)\vec{r};\\
&L_{i5}:\ \vec{M}=\vec{A}+\left(\frac{G}{r}\right)\vec{r};\quad L_{i6}:\ \vec{\Gamma}=r\vec{p}+\left(\frac{G}{rG'}-1\right)\vec{r}K-\frac{iGG''}{2rG'^2}\vec{r};\\
&L_{56}:\ \Gamma_0=\frac{G}{2G'^2}K^2+\frac{L_0{}^2}{2G}-\frac{GG'''}{4G'^3}+\frac{3GG''^2}{8G'^4}+\frac{G}{2};\quad L_{46}:\ \Gamma_4=\Gamma_0-G;\quad L_{45}:\ T=\frac{G}{G'}K-i\frac{GG''}{2G'^2}.
\end{aligned} \tag{1}$$

Here $\vec{L}_0$ and $\vec{A}$ generate the subgroup SO(4), $\vec{L}_0$ and $\vec{M}$ the Lorentz subgroup SO(3, 1), $(\Gamma_0, \Gamma_4, T)$ are the generators of the radial SO(2, 1) dynamical group, and $\Gamma_\mu = (\Gamma_0, \vec{\Gamma})$ is a vector-current operator with respect to the Lorentz group. Further $K$ is the Hermitian radial momentum operator of the relative motion,

$$K = \tfrac{1}{2}(\hat{r}\cdot\vec{p} + \vec{p}\cdot\hat{r}) = r^{-1}(\vec{r}\cdot\vec{p} - i). \tag{2}$$

If the arbitrary function $G(r)$ is chosen to be $G(r) = r$, we get the well-known realization of O(4, 2) appropriate for the Kepler problem. The new cases are $G = r^2/2$ appropriate for the oscillator problem, and $G = e^{-ar}$ for the problem of Morse oscillator ($S$ wave). The unified forms of the radial algebra $SO(2,1) = (\Gamma_0, \Gamma_4, T)$ for these cases have been known,[5] but not the full relativistic group SO(4, 2). The Hamiltonian in these problems is in the Lie algebra of SO(2, 1), an even simpler case than the H atom, where not $H$ but $r(H-E)$ is a linear combination of $\Gamma_0$ and $\Gamma_4$. The general case $G(r)$ can be considered to be a canonical transformation from the case $G(r) = r$ of the coordinates:

$$\vec{r} \to G(r)\vec{r}/r \equiv \vec{r}\,',$$

$$\vec{p} \to \frac{r}{G}\vec{p} + \left(\frac{1}{rG'} - \frac{1}{G}\right)\vec{r}K + i\,\frac{G''\vec{r}}{2rG'^2} \equiv \vec{p}\,'.$$

This generalizes the canonical transformation of the radial equation between oscillator and Kepler potential.[6] The representation (1) is the restriction of a covariant representation of O(4, 2) on a three-dimensional surface, and has a dual in which $\vec{r} \to \vec{p}$, $\vec{p} \to -\vec{r}$.[7]

According to the general theory of relativistic composite objects based on dynamical groups[8] we can write a covariant Dirac-type wave equation of the form

$$(J^\mu P_\mu + \beta\Gamma_4 + \gamma)\psi(P) = 0, \tag{3}$$

where $P_\mu$ is the (total) momentum of the moving system and the general form of the current $J^\mu$ linear in momentum and in the generators is

$$J_\mu = \Gamma_\mu + \alpha_2 P_\mu + \alpha_3 P_\mu \Gamma_4, \tag{4}$$

with constants $\beta$, $\gamma$, $\alpha_2$, $\alpha_3$. The relativistic H atom is well described by such an equation including the recoil corrections to the spectrum,[8] form factors,[9] and structure functions.[10]

For the oscillator the equation is an even simpler special case of (3). With $\beta = \alpha_3 = 0$, $\alpha_2 = -1/\lambda$,

$$(\Gamma^\mu P_\mu - \lambda^{-1} P_\mu P^\mu + \gamma)\psi = 0. \tag{5}$$

The term $\alpha_3$ in the current gives rise to accumulation of the discrete spectrum and a continuum as in the H atom. Here $\Gamma_\mu$ is a differential operator in the canonical coordinates $(\vec{r}, \vec{p})$ only, i.e., the internal motion. It can also be written in terms of the creation and annihilation operators $a_i, a_i^\dagger$ of the oscillator. We now show that in the rest frame Eq. (5) coincides with the Hamiltonian of the oscillator.

With $P_\mu = (M, \vec{0})$ the rest-frame dynamics is given by

$$(\Gamma_0 M - \lambda^{-1}M^2 + \gamma)\psi = 0,$$

and for $\gamma = 0$,

$$\lambda\Gamma_0\psi = M\psi. \tag{6}$$

The spectrum of $\Gamma_0$ is known from the Casimir operator of the SO(2, 1) which is $l_0(l_0+1)$:

$$\mathrm{Spec}\,\Gamma_0 = s + l_0 + 1; \quad s = 0, 1, 2, \ldots . \tag{7}$$

On the other hand, for the oscillator $(G = \tfrac{1}{2}r^2)$, $\Gamma_0$ reduces to the operator

$$\Gamma_0 = \frac{1}{4}\left(K^2 + \frac{4L_0^2 + \frac{3}{4}}{r^2} + r^2\right) = \frac{E}{2\hbar\omega}, \tag{8}$$

where $E$ is the oscillator energy. We see from (8) that the total angular momentum of the oscillator $\vec{L}$ is related to the group generator $\vec{L}_0$ by

$$L^2 = 4L_0^2 + \tfrac{3}{4}. \tag{9}$$

Hence, $l(l+1) = 4l_0(l_0+1) + \frac{3}{4}$, or $l = 2l_0 + \frac{1}{2}$. Thus expressing the spectrum of $\Gamma_0$ in (7) in terms of $l$, we have the mass spectrum

$$M_n = \tfrac{1}{4}\lambda[n \pm (n^2 + 4\gamma/\lambda)^{1/2}], \quad n = \tfrac{1}{2}(2s + l + \tfrac{3}{2}),$$

$$M_n = \tfrac{1}{2}\lambda(2s + l + \tfrac{3}{2}), \quad s = 0, 1, 2, \ldots . \tag{10}$$

The complete solution of (5) is obtained by a boost generated by $L_{i5}$ given in (1). The degeneracy of the levels is exactly that of the oscillator and the meson spectrum is consistent with this degeneracy as shown in Fig. 1. (For the H atom all the vertices in the diagram would be filled.) Equation (9) indicates that although we have SU(3) symmetry in the quantum number $n$, we have O(4) symmetry in the quantum number $(n+l)$, just like the Madelung quantum number of the periodic table.

We remark that the coordinates $\vec{r}, \vec{p}$ are just effective labels for the internal dynamics, just as in the case of Dirac's new wave equation with a two-dimensional internal dynamics.[11]

The wave equation (3) is a special case of a class of infinite-component wave equations de-

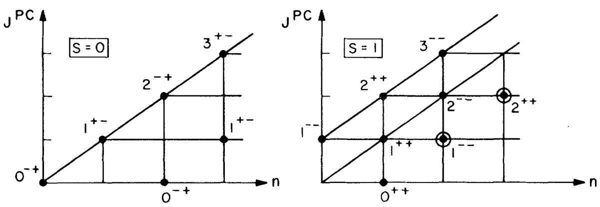


FIG. 1. The $(n,J)$ diagram for the three-dimensional oscillator levels (black dots) and comparison with the meson levels. In a $(q\bar{q})$ model for $S=0$, $J=l$. For a hydrogenic spectrum all corners of the diagram are filled. For $S=1$, we take the additional degeneracy coming from $l=J-1$, $J$, $J+1$.

scribing relativistic composite objects for which the form factors and structure functions have been extensively studied, when the equation is coupled to an external field by minimal substitution. We now see that the special case (5) is actually realized by the relativistic moving oscillator. In our model of two spinless constituents, the electric form factor is given by

$$G_E(t)=\frac{1}{(1-a^2t)}+\frac{2Ma}{(1-a^2t)^2},$$

and hence dominated by a single pole. The structure functions appearing in deep inelastic scattering are also known explicitly.[10] In the scaling limit one obtains

$$W_1(Q^2,\nu)=O(1/\nu)\to 0,$$
$$\nu W_2(Q^2,\nu)=\xi^2(1-\xi),$$

where $Q^2$ is the momentum transfer, $\nu$ is the energy variable, and $Q^2=\xi(M^2+2M\nu)$. The threshold behavior of $W_2$ for $\xi\to 1$ is consistent with the single power law in the form factor $G_E(t)$ according to the general arguments.[12]

[1]R. O. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D 3, 2706 (1971

[2]L. P. Hughston, Proc. Roy. Soc. London, Sect. A 382, 459 (1982).

[3]For example, H. van Dam and L. C. Biedenharn, Phys. Lett. 8B, 313 (1979), and references therein.

[4]A. O. Barut, *Dynamical Groups and Generalized Symmetrics in Quantum Theory* (Univ. of Canterbury Press, Christchurch, New Zealand, 1972).

[5]A. O. Barut, H. Beker, and D. Dibekci, J. Bogazici Univ., 8-9, 11-33 (1980-81).

[6]A. O. Barut and J. H. Duru, Proc. Roy. Soc. London, Sect. A 333, 217 (1973); N. Rowley, J. Phys. A 12, L7 (1979); M. Moshinsky and T. H. Seligman, J. Math. Phys. (N.Y.) 22, 1527 (1981).

[7]A. O. Barut and H. Beker, unpublished.

[8]A. O. Barut and W. Rasmussen, J. Phys. B 6, 1695, 1713 (1973).

[9]A. O. Barut and R. Wilson, Phys. Rev. A 13, 918 (1976).

[10]A. O. Barut and R. Wilson, Phys. Rev. D 13, 2629, 2647 (1976).

[11]P. A. M. Dirac, Proc. Roy. Soc. London, Sect. A 322, 435 (1971), and 328, 1 (1972).

[12]C. G. Callan and D. J. Gross, Phys. Rev. Lett. 22, 156 (1969); S. D. Drell and T.-M. Yan, Phys. Rev. Lett. 24, 181 (1970).