One-Dimensional Ising Model in a Random Field

R. Bruinsma
IBM T. J. Watson Research Center, Yorktown Heights, New York 10598

and

G. Aeppli
Bell Laboratories, Murray Hill, New Jersey 07974
(Received 14 January 1983)

The one-dimensional Ising model in a random field is studied with use of a functional recursion relation. For temperatures exceeding a given value, the fixed function of the relation is found and shown to be a devil's staircase. From this result it is possible to evaluate the free energy to arbitrary precision. In the field-strength–temperature plane, a crossover line corresponding to the onset of frustration is found.

PACS numbers: 75.40.Dy, 05.50.+q

Quenched impurities in magnets can cause randomness in the exchange interactions and local moments. A uniform magnetic field applied to a disordered magnet can also induce local random fields, conjugate to the order parameter.[1] These random fields destroy, according to both theory[2] and experiment,[3] long-range magnetic order for spatial dimensionalities d less than some critical d_c. There has been considerable controversy[4] about whether d_c is 2 or 3 for Ising magnets and more generally about the relative importance of thermal fluctuations and quenched randomness. Because there are no exact results for $T \neq 0$, even for $d = 1$, we have studied the finite temperature properties of the one-dimensional Ising ferromagnet in a binary random field ($\pm h_0$). This problem is equivalent to the mixed ferromagnetic and antiferromagnetic chain (a $d=1$ spin-glass) in a uniform magnetic field. It has been studied numerically[5] for general T and analytically[6,7] for $T \to 0$. Such systems can be realized experimentally in mixtures of quasi one-dimensional magnetic materials.[8]

Our results are best described in terms of the local magnetization $m = \langle S_i \rangle$, which can be probed with use of nuclear magnetic resonance or Mössbauer spectroscopy. For large ratios of the random-field amplitude h_0 to the exchange coupling J_0, or high temperatures (region I of Fig. 1), every spin S_i follows its local random field. Furthermore, m takes on only *discrete* values and so its integrated probability distribution $Q(m)$ is flat nearly everywhere; indeed $Q(m)$ is an example of a nonanalytic "devil's staircase" function (Fig. 2). As h_0 is reduced, we cross a transition line into region II where m can take

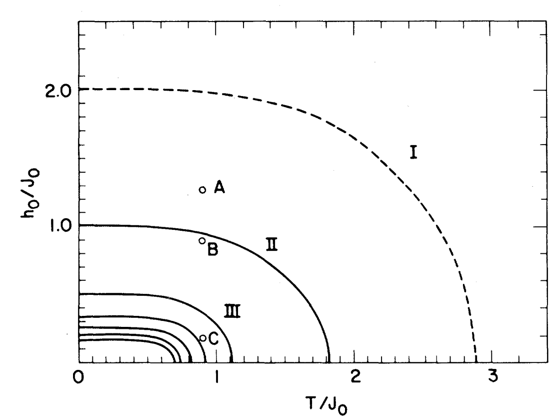

FIG. 1. Phase diagram in h_0/J_0-T/J_0 plane. The solid lines are transition lines for $P(z)$. $P(z)$ is a devil's staircase in regions I and II. The dashed line corresponds to onset of frustration. $P(z)$ for points A, B, and C is shown in Fig. 2.

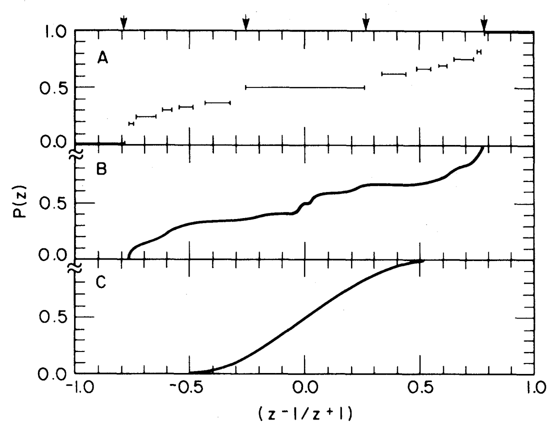

FIG. 2. $P(z)$ for $J = 1.1$, from Eq. (11) for (a) $h = 1.4$, and (b) from iterated evaluation ($n = 1$ to $n = 40$) of Eq. (9) for $h = 1$ and (c) $h = 0.2$. We use $g(z) = (z-1)/(z+1)$ rather than z as ordinate. From left to right, the arrows in (a) correspond to z_L, $z_L(\frac{1}{2})$, $z_U(\frac{1}{2})$, and z_U, respectively.

on values in a *continuum, which includes zero.* If $m=\langle S_i\rangle=0$, then the local random field h_i is precisely cancelled by the effective field due to the neighboring spins S_{i-1} and S_{i+1}, and so S_i is *frustrated.* For $T\to 0$ the transition line corresponding to the onset of frustration terminates at $h_0/J_0=2$ which is also where the ground-state entropy first becomes nonzero. As we continue to decrease h_0 and T we find an infinite series of further transition lines related to increasing smoothness of $Q(m)$. Eventually for $T\ll J_0$ and $h_0\lesssim T\exp(-2J_0/T)$, $Q(m)$ becomes indistinguishable from a Gaussian distribution.

The starting point is the Hamiltonian

$$\mathcal{H}/T=-J\sum_{i=1}^{N} S_iS_{i+1}+\sum_{i=1}^{N} h_iS_i\,, \tag{1}$$

where $J=J_0/T$ and where $S_i=\pm 1$ is the spin at site i. The random field variables h_i are uncorrelated and have the same distribution function $\rho(h_i)$. The partition function $Z\equiv\sum_{\{s\}}e^{-\mathcal{H}}$ can be written as

$$Z=(1\ 1)\left(\prod_{i=1}^{N}\hat{T}_i\right)\begin{pmatrix}1\\1\end{pmatrix}, \tag{2}$$

where $\hat{T}_i$ is a 2×2 random matrix,

$$\hat{T}_i=\begin{bmatrix}\exp(J+h_i) & \exp(-J-h_i)\\ \exp(-J+h_i) & \exp(J-h_i)\end{bmatrix}. \tag{3}$$

To evaluate Z we follow Brandt and Gross[6] and study the evolution of a two-component vector on successive application of $\hat{T}_i$:

$$\vec{v}_n=\begin{pmatrix}v_n^{+}\\ v_n^{-}\end{pmatrix}=\hat{T}_n\cdot\hat{T}_{n-1}\cdots\hat{T}_1\begin{pmatrix}1\\1\end{pmatrix}. \tag{4}$$

Equation (4) yields a recursion relation for the tangent z_n of the angle between $\vec{v}_n$ and $\binom{1}{0}$,

$$z_n=f(h,z_{n+1})\,, \tag{5}$$

where

$$f(h,z)=e^{2h}\left(\frac{ze^{J}-e^{-J}}{e^{J}-ze^{-J}}\right). \tag{6}$$

For a pure system, $h_n=h$ and z_n is uniquely defined. As $n\to\infty$, z_n approaches the simple fixed point

$$z(h)=f(h,z(h))=-\exp(2J+h)\sinh h+e^{h}(1+e^{4J}\sinh^2 h)^{1/2}. \tag{7}$$

For a random system, z_n will itself be a random variable whose integrated distribution function is the quenched average

$$P_n(z)=\prod_{i=1}^{N}\int_{-\infty}^{+\infty}dh_i\,\rho(h_i)\,\theta(z-z_n)\,, \tag{8}$$

where $\theta(z)$ is the Heaviside step function. From Eq. (5) we obtain the recursion relation

$$P_{n+1}(z)=\int_{-\infty}^{+\infty}dh\,\rho(h)P_n(f(h,z))\,. \tag{9}$$

If the collection of fixed points $z\equiv\lim_{n\to\infty}z_n$ has a well-defined integrated distribution $P(z)$, then $P(z)$ must be a fixed point of the functional recursion relation Eq. (9).

For the remainder of this paper, we restrict our attention to discrete random fields where $h_i=+h=h_0/T$ and $h_i=-h=-h_0/T$ each occur with probability $\frac{1}{2}$. In this case $P(z)$ satisfies

$$P(z)=\tfrac{1}{2}\left\{P\left(e^{2h}\left[\frac{e^{J}z-e^{-J}}{e^{J}-ze^{-J}}\right]\right)+P\left(e^{-2h}\left[\frac{e^{J}z-e^{-J}}{e^{J}-ze^{-J}}\right]\right)\right\}. \tag{10}$$

Now we construct[9] the solution to Eq. (10). Since the z_n are nonnegative, $P(z)$ vanishes for $z<0$. Because $P(z)$ is monotonically increasing and $f(\pm h,z)$ has a zero at $z=e^{-2J}$, it follows that $P(z)=0$ for $z\leqslant e^{-2J}$. Repetition of this argument implies that $P(z)=0$ as long as $z>f(\pm h,z)$. Since $f(h,z)>f(-h,z)$ for $e^{-2J}<z<e^{2J}$, $P(z)=0$ for $z<z_L$ where $z_L=f(h,z_L)$ or $z_L=z(h)$ [see Eq. (7)]. Similarly, $P(z)=1$ if $z>z_U$ where $z_U=z(-h)$. Now, the first term in Eq. (10) is 1 for $z>z_L(\frac{1}{2})$ where $f(h,z_L(1/2))=z_U$, while the second term is 0 if $z<z_U(\frac{1}{2})$ where $f(-h,z_U(1/2))=z_L$, so $P(z)=\frac{1}{2}$ if $z_L(\frac{1}{2})<z<z_U(\frac{1}{2})$ [Fig. 2(a)]. The values of $z_U(\frac{1}{2})$ and $z_L(\frac{1}{2})$ are easily derived from Eqs. (6) and (7). Repeating this argument, we find that at the mth iteration $P(z)=(2p-1)/2^m$ for

$$z_L([2p-1]/2^m)<z<z_U([2p-1]/2^m)\,,$$

where

$$z_L([2p-1]/2^m)=f(h,z_L([2p-1]/2^{m-1}))\,,$$
$$z_U([2p-1]/2^m)=f(h,z_U([2p-1]/2^{m-1}))\,, \tag{11a}$$

for $2p-1<2^{m-1}$, and

$$z_L([2p-1]/2^m)=f(-h,z_L([2p-1]/2^{m-1}-1)),$$
$$\tag{11b}$$
$$z_U([2p-1]/2^m)=f(-h,z_U([2p-1]/2^{m-1}-1))\,,$$

for $2p-1>2^{m-1}$. Equations (11a) and (11b) give $P(z)$ to arbitrary precision. As $m\to\infty$, $P(z)$ becomes a devil's staircase [Fig. 2(a)]: an infinite series of rising steps, each of finite width. The widest step is for $P(z)=\frac{1}{2}$. A necessary and sufficient condition for every step to be of finite width is that $z_L(\frac{1}{2})<z_U(\frac{1}{2})$. This condition is first violated when $f(h,1)=z_U$ or

$$e^{2J}=1+2\cosh(2h). \tag{12}$$

Equation (12) defines a line which divides the h/J-J^{-1} (or T) plane into a small-J, large-h regime where $P(z)$ is a devil's staircase, and a large-J, small-h region where we expect $P(z)$ to be smooth [Fig. 2(b)]. This crossover line separates regions II and III in Fig. 1; it intersects the J axis ($h=0$) at $J=\frac{1}{2}\ln 3$, and approaches the asymptote $J=h$ in the limit $T\to 0$ where $J,h\to\infty$.

Even though we cannot write down the solution to Eq. (10) when $z_L(\frac{1}{2})>z_U(\frac{1}{2})$, it is still possible to find $P(z)$ near certain z. In particular, as $z\to z_U$

$$P(z)=1-c(z_U-z)^{\ln 2\,\xi_{\rm FM}}, \tag{13}$$

where c is a positive constant and $\xi_{\rm FM}$ the correlation length of the pure Ising ferromagnet in a uniform field h. Equation (13) may be checked by substitution into Eq. (10). A similar result holds near z_L.

Consider now the functional equation obtained after application of Eq. (10) to the terms on its right-hand side. If $f(h,f(h,z))>z_U$ and $f(-h,f(-h,z))<z_L$ this equation reduces to

$$P(z)=\tfrac{1}{4}\{P\big(f(h,f(-h,z))\big)+P\big(f(-h,f(h,z))\big)\}+\tfrac{1}{4}. \tag{14}$$

Equation (14) implies that where $f(-h,f(h,z_1))=z_1$ and $f(h,f(-h,z_2))=z_2$, $P(z_1)=\frac{1}{3}$ and $P(z_2)=\frac{2}{3}$. The bistable fixed points z_1 and z_2 correspond to an Ising ferromagnet in a staggered field, or equivalently, an antiferromagnet in a dc field of strength h. Letting $\xi_{\rm AFM}$ denote the correlation length of these pure systems, we find that near z_1 and z_2, $P(z)$ has a power-law singularity, as in Eq. (13), with exponent $(2\ln 2)\xi_{\rm AFM}$.

Equation (14) becomes invalid when $f(h,f(h,1))=z_U$. For these values of J and h, the behavior of $P(z)$ near $z=1$ is similar to that at z_U and z_L:

$$P(z)\cong\tfrac{1}{2}+c'\,\mathrm{sgn}(z-1)|z-1|^{(\ln 2)\xi_{\rm FM}}. \tag{15}$$

We can also iterate Eq. (10) a second time, which yields $P(z)=\frac{1}{7},\frac{2}{7},\ldots,\frac{6}{7}$ at the tristable fixed points corresponding to the pure Ising ferromagnet in a staggered field of period three. In general, if

$$f\big(h,f(h,\ldots,f(h,1)),\ldots\big)\equiv f^{(n)}(J,h)=z_U, \tag{16}$$

$P(z)$ will take the form (15) near $z=1$. Furthermore, for at least some J and h such that $f^{(n)}(J,h)>z_U$, $P(z)$ can be calculated at the n-stable fixed points characterizing the pure chain in a staggered field of period n. Condition (16) defines a family of curves in the h/J-J^{-1} plane (solid lines in Fig. 1); these curves terminate at the $J\to\infty$ (or $T\to 0$) phase transition points[7] where J/h is an integer m. However, we do not recover the phase transitions found by Derrida and co-workers[7] for $2J/h$ assuming odd integer values. In the limit $h\to 0$, our curves meet the J axis ($h=0$) when $\xi_{\rm FM}=[\ln(\coth J)]^{-1}=m/\ln 2$, a result confirmed by linear-response theory.[10] As $h\to 0$ and $J\to\infty$, with he^{2J} finite, $P(z)$ becomes progressively smoother [compare Figs. 2(b) and 2(c)] and (10) eventually reduces to the differential equation

$$[\coth J-1](z-1)P'(z)+2h^2P''(z)=0, \tag{17}$$

whose solution is the integral of a Gaussian with standard deviation $(2h)^{1/2}(\coth J-1)^{1/2}$.

We have established the angular distribution of the vectors $\vec{v}_n$ as $n\to\infty$. We still wish to know how, for a given $z_n=\tan\theta_n$, the length $|\vec{v}_n|$ is distributed. Because $|\vec{v}_n|\to\infty$, as $n\to\infty$, it is useful to introduce the variable

$$y_n=-(2n)^{-1}\ln(v_n^{\ +}+v_n^{\ -}). \tag{18}$$

Using recursion relations similar to those for z_n we find that as $n\to\infty$, the probability distribution of y_n becomes, for a given z, a delta function centered at $F(z)$ where

$$F(z)=-\tfrac{1}{4}\ln[2\cosh 2J+ze^{2h}+z^{-1}e^{-2h}]-\tfrac{1}{4}\ln[2\cosh 2J+ze^{-2h}+z^{-1}e^{2h}]. \tag{19}$$

We obtain the quenched average of the free energy from Eqs. (2), (4), (18), and (19):

$$F=\int_0^\infty F(z)[dP(z)/dz]dz, \tag{20}$$

$$=F(z_U)-\sum_{m=1}^{\infty}\sum_{p=1}^{2^{m-1}}\{F(z_U([2p-1]/2^m))-F(z_L([2p-1]/2^m))\}. \tag{21}$$

Far from the crossover line, the series converges rapidly and it is a good approximation to keep only the first term. In the limit $T \to 0$, where h and J are large, the crossover point is at $h/J = 1$. For $h/J > 1$, keeping only the lowest term gives

$$F \simeq -\tfrac{1}{2}(J+h) - \tfrac{1}{4}\ln[e^{2J} + e^{2h-2J}]\,. \tag{22}$$

If $T \to 0$, $F = -h$ for $h/J > 2$ and $F = -J - \frac{1}{2}h$ for $1 < h/J < 2$. Thus we recover the first of the $T \to 0$ phase transitions.[7] At low but finite temperatures, the transition is smeared out.

We can also use $P(z)$ to determine the local magnetization distribution. Namely,

$$m = \langle S_i \rangle = \frac{z_1 z_2 - \exp(-2h_i)}{z_1 z_2 + \exp(-2h_i)}\,, \tag{23}$$

where z_1 and z_2 are independent random variables, both with integrated probability distribution $P(z)$. Equation (23) implies that the probability density dQ/dm for m vanishes at $m = \langle S_i \rangle = 0$ when

$$e^{2J} < 2\cosh h\,. \tag{24}$$

As $h \to 0$, $Q(m)$ is a devil's staircase if condition (24) is satisfied,[10] and numerical work suggests that $Q(m)$ remains a devil's staircase for large h as well. Once (24) is violated, $\langle S_i \rangle$ is allowed to be zero and the threshold line defined by (24) thus corresponds to the onset of frustration. Indeed, as $T \to 0$, this line approaches $2J/h = 1$, which is where the ground-state entropy first becomes nonzero due to frustration.

We thank A. Aharony, D. Andelman, M. Azbel, R. Birgeneau, V. Emery, B. Halperin, P. Horn, M. Ma, D. Mukamel, and D. Stone for useful discussion. We are especially grateful to J. Fernandez for giving us the results of a numerical simulation, and to T. Schultz for a careful reading of the manuscript. This work was begun at Brookhaven National Laboratory and partially supported under U. S. Department of Energy Contract No. DE-AC02-76CH00016.

[1]S. Fishman and A. Aharony, J. Phys. C 12, L729 (1979).

[2]Y. Imry and S.-k. Ma, Phys. Rev. Lett. 35, 1399 (1975).

[3]D. B. Belanger, A. R. King, and V. Jaccarino, Phys. Rev. Lett. 48, 1050 (1982), and references therein.

[4]G. Grinstein and S.-k. Ma, Phys. Rev. Lett. 49, 685 (1982), and references therein.

[5]M. Puma and J. F. Fernandez, Phys. Rev. B 18, 1391 (1978); M. Azbel and M. Rubinstein, to be published.

[6]U. Brandt and W. Gross, Z. Phys. B 31, 237 (1978).

[7]B. Derrida, J. Vanninemus, and Y. Pomeau, J. Phys. C 11, 4749 (1978).

[8]P. Horn, private communication.

[9]For a similar case: E. Lieb and D. Mattis, *Mathematical Physics in One Dimension* (Academic, New York, 1966), p. 123.

[10]G. Aeppli and R. Bruinsma, to be published.