Threshold Behavior of Charge-Density Waves Pinned by Impurities

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A time-dependent mean-field theory of the nonlinear electric field response of a charge-density wave pinned by impurities is implemented. It is found that above a threshold electric field E_T, the charge-density wave moves with velocity $v \propto (E-E_T)^{3/2}$. Some general discussion of the behavior as a function of dimension is included.

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In electric fields E above a threshold field E_T, the incommensurate charge-density wave (CDW) in $NbSe_3$ is observed to move and contribute to the nonlinear conductivity.[1-3] This threshold is believed to be due to pinning of the CDW by impurities.[4] Several attempts have been made recently to explain various aspects of the nonlinear response of this system. However, the behavior near threshold, in particular the dependence of the nonlinear current or CDW velocity v on $E - E_T$, is not yet understood.[4-7] Grüner, Zawadowski, and Chaikin[6] treat the whole of the CDW as a macroscopic rigid classical object moving in an effective sinusoidal potential. While this appears to be a convenient way to analyze aspects of the data in some regimes, it is far from being derived from a reasonable microscopic or semimicroscopic model with elastic internal degrees of freedom. Bardeen[5] has proposed that macroscopic sections of the CDW tunnel quantum mechanically as rigid objects, again ignoring the internal degrees of freedom of the CDW.

Analysis of a microscopic model of the CDW as a classical, deformable medium interacting with a weak random potential caused by impurities[4] was carried out by Sneddon, Cross, and Fisher.[7] The pinning of the CDW is caused by *collective* effects of the impurities. Agreement[7] with experiment was found for fields well above threshold for both the nonlinear high-field conductivity[1] (which does *not* agree with predictions of Refs. 5 or 6) and ac-dc interference effects.[8] This model, however, is very difficult to analyze in the interesting regime near threshold because of the existence of a correlation length ξ which diverges as the velocity of the CDW goes to zero. This correlation length measures the size of regions of the CDW in which there is phase coherence, i.e., which move as almost rigid, though deformed, objects. The behavior near threshold is thus a critical phenomenon and merits treatment as such.

In this paper a simplified version of the model of Ref. 7 is studied in mean-field theory and shown to yield a threshold behavior $v \propto (E - E_T)^{3/2}$. This threshold behavior occurs in mean-field theory only if the strength of pinning, which is characterized by a parameter D, is greater than a critical value D_M at which metastable states first appear in zero electric field. For $D < D_M$ there is no threshold and $v \propto E$ for small E. The mean-field behavior as a function of D and E is illustrated in Fig. 1(a). General discussion of the behavior and dimensionality dependence of CDW's interacting with impurities is contained at the end, along with some comments concerning the experiments.

Let us consider a simple model of a d-dimensional, single-wave-vector, charge-density wave[4] interacting with impurities at positions $\{\vec{R}_j\}$; each tries to pin the local phase φ_j at β_j with the $\{\beta_j\}$ randomly distributed between 0 and 2π. The Hamiltonian is

$$H = \frac{1}{4}\sum_{ij} J(\vec{R}_i - \vec{R}_j)(\varphi_i - \varphi_j)^2 - \sum_j h_j \cos(\varphi_j - \beta_j), \quad (1)$$

where the first term with $J(\vec{R})$ short ranged represents the elasticity of the charge density wave which favors uniform phase. The $h_j > 0$ are random impurity strengths independently distributed with probability $P(h)$ which is taken to be 0 for h greater than a value D which characterizes the distribution. By assuming that the elastic interaction in Eq. (1) is of the form $\frac{1}{2}(\varphi_i - \varphi_j)^2$ [rather than, e.g., $1 - \cos(\varphi_i - \varphi_j)$], we have ignored phase-slip processes due to defects in the CDW.

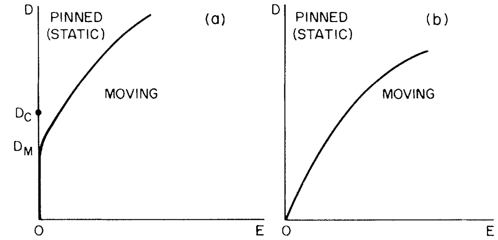

FIG. 1. Schematic phase diagram as a function of the maximum pinning potential D and electric field E for (a) $d > 4$ and (b) $d < 4$. The phase boundary is the threshold field $E_T(D)$.

The inclusion of defects (e.g., dislocations) considerably complicates the results and will not be discussed here. Their neglect is justified if the CDW is stiff and the impurities far apart.

We will take purely relaxational equations of motion for the phases: $d\varphi_j/dt = -\delta H/\delta\varphi_j + E$, where E simulates an applied electric field in the direction of the CDW wave vector.

At high electric fields, $E \gg D$, the average phase $\overline{\varphi}(t)$ advances with velocity v: $\overline{\varphi}(t) = vt$, where $v = E - O(D^2/E)$. In this regime, the deviations of the local phase from $\overline{\varphi}$ are small and can be treated perturbatively (analogously to Ref. 7). However, in the interesting regime $E \sim E_T$ these deviations become large and occur on all length scales. Truncation of the model to just *one* impurity (as in Ref. 6 with h somehow describing the overall effects of all the impurities) enables it to be solved simply yielding a threshold field $E_T = h$ with $v = 0$ for $E < E_T$ and $v \sim (E - E_T)^\zeta$ for $E > E_T$ with $\zeta = \frac{1}{2}$. Straightforward arguments show[9] that ζ equals $\frac{1}{2}$ quite generally for any *finite* number N_I of $\{\varphi_j\}$. We are, however, interested in the limit $N_I \to \infty$.

In this paper we make the simplest approximation to the general problem by carrying out a time-dependent mean-field theory. This is equivalent to taking infinite-range interactions, i.e., *all* $J(\vec{R}) = J/N_I$ with $N_I \to \infty$ where the $1/N_I$ is necessary to keep the overall interaction strength finite. The equation for a single phase involves the average over all others which for $N_I \to \infty$ we replace by its smooth time-dependent average $\overline{\varphi}(t)$.

Each phase will fluctuate around the average phase and the pinning is due to the collective effects of these fluctuations. The mean-field equation of motion for a phase φ_j is

$$d\varphi_j/dt = -h_j \sin(\varphi_j - \beta_j) + E + \overline{\varphi}(t) - \varphi_j, \qquad (2)$$

where for convenience we have set $J = 1$. Self-consistency requires that the solutions to the set of equations for the $\{\varphi_j\}$ satisfy $N_I^{-1}\sum_j \varphi_j(t) = \overline{\varphi}(t)$. Details will be presented elsewhere and we restrict ourselves here to the results and a discussion of the important features of the solution.

The equation of motion for φ_j can be obtained from a time-dependent potential $\Phi_j(\varphi_j, t)$ by $d\varphi_j/dt = -\delta\Phi_j/\delta\varphi_j$. Two cases must now be distinguished: (i) Weak pinning: $D < D_M = 1$ (i.e., *all* $h_j < 1$). In this case, for each phase the potential $\Phi_j(\varphi_j)$ will have only *one* minimum at all times. (ii) Strong pinning: $D > D_M$. For $h_j > 1$, $\Phi_j(\varphi_j)$ will have two (or more) minima for at least some t.

(i) *Weak Pinning.*—We first consider *static* solutions, i.e., $d\varphi_j/dt = 0$ for all j. For this case self-consistent static solutions only exist if $E = 0$ and they are unique up to the overall average phase $\overline{\varphi}$. For weak pinning there is thus *no* threshold field. We thus turn to the dynamics and search for *uniformly moving* steady-state solutions with $\overline{\varphi} = vt$. We assume that all transients have died away and each phase is moving such that $\varphi_j - vt$ is periodic in time with period $T = 2\pi/v$. In the absence of a threshold field, there is a linear response at small E, i.e., $v = \sigma_0 E$ for $E \to 0$. However, the proportionally constant, σ_0, the *linear* CDW conductivity, is less than the general high-field value of 1 (see discussion above) and is given by

$$\sigma_0 = \left[\int_0^D P(h)(1-h^2)^{-1/2}dh\right]^{-1}. \qquad (3)$$

Note that as long as $P(h)$ is finite as $h \to D$, $\sigma_0 \to$ constant as $D \to 1^-$. Thus, in the weak-pinning limit the conductivity is nonlinear but has a linear regime for small fields.

In the limit $v \to 0$, each phase φ_j only deviates by an amount of order v from the (time-dependent) minimum of its effective potential Φ_j. Equation (4) is derived from an expansion of φ_j about its value at this minimum. We now turn to the more interesting strong-pinning case.

(ii) *Strong pinning.*—We again first consider static solutions. Because of the existence of several minima of $\Phi_j(\varphi_j)$ for those j with $h_j > 1$, many self-consistent, linearly stable, static configurations are possible for E less than a threshold field $E_T > 0$. However, as $E \to E_T^-$ it can be shown that these solutions become more similar until at $E = E_T$ there is a *unique* (up to an overall phase) self-consistent solution and for $E > E_T$ no static solutions exist. For $D \to 1^+$, the threshold field is given by $E_T = \int_1^D (9/4\pi)(h-1)^2 P(h)dh$, and hence tends to 0 as $D \to 1^+$.

We note that below threshold hysteretic behavior exists. If the electric field is increased and decreased adiabatically, the resulting configuration will not be the same as the initial state. The linear response to an ac field below threshold and other properties in this regime will be discussed in a longer paper.

A steady-state *moving* solution exists in the strong-pinning limit only for $E > E_T$ and, like the static solution at $E = E_T$, is *unique* (up to a trivial overall phase). In contrast to the weak-pinning case, the motion of each phase for strong pinning is rather complicated at low velocities. This is due to the emergence, time evolution, and disappearance of minima of some of the time-dependent potentials $\Phi_j(t)$. For those φ_j with $h_j > 1$ the motion is as follows: For most of each period, φ_j lags behind the smallest φ_j minimum of Φ_j by a small amount $[O(v)]$ and moves with a velocity of order v. However once each period, T, this minimum disappears and in a short transit time which turns out to be order $v^{-1/3}$, φ_j moves rapidly to catch up with the next minimum. A careful asymptotic analysis of this behavior and the mean-field self-consistency condition yields the velocity near threshold,

$$v = B(E - E_T)^{3/2} + O((E - E_T)^2 \ln(E - E_T)), \qquad (4)$$

i.e., $\zeta = \frac{3}{2}$, the main result of this paper. For $D \to 1^+$, the boundary between weak and strong pinning, the coefficient B has the form

$$B = [b \int_1^D (h-1)^{1/3} P(h) dh]^{-3/2}, \qquad (5)$$

where $b = 3y_0 2^{-1/3}/\pi$ with y_0 the smallest zero of the Airy function $\mathrm{Ai}(-y)$. Therefore, $B \to \infty$ as $D \to 1^+$, with the divergence dependent on the form of $P(h)$. In this limit, the range of applicability of Eq. (4) will approach zero—characteristic of the behavior near a multicritical point. Note that for high fields $E \gg D$, $v = E - (2E)^{-1} \int_0^D h^2 P(h) dh$ for both strong and weak pinning.

In Fig. 1(a), a mean-field "phase" diagram is shown as a function of D and E; note the multicritical point at $E = 0$, $D_M = 1$.

By analogy with critical phenomena, one expects mean-field theory to be valid near the threshold field E_T, for spatial dimensions d greater than an upper critical dimension d_c. It is not at all clear at this stage, however, what d_c is. For $d < d_c$, ζ will change but there will be a threshold field in *any* dimension for sufficiently large impurity pinning strengths and ζ will always be defined, in contrast to usual critical phenomena. Note that even in $d = 0$, which corresponds to a finite number of $\{\varphi_j\}$, $E_T > 0$ and $\zeta = \frac{1}{2}$ as noted above.

I now speculate on the form of the phase diagram as a function of D and E, for various d.[10] For $d > 4$ there will exist a weak-pinning regime at small D with no metastable states, no threshold, and linear response [as above and in Fig. 1(a)]. This will be separated by a multicritical point D_M from the strong-pinning regime with metastable states and a threshold. In $d > 4$ there will also be some $D_c > D_M$ such that the *ground* state at $E = 0$ will have long-range order only for $D < D_c$.[11] However, we are concerned here with metastable states which will generally *not* have long-range order except for $D < D_M$ where there is a unique state. For $E > E_T(D)$, in all dimensions, I believe that there will be a unique state which will have time-dependent long-range order, with a well-defined average phase $\bar{\varphi}(t) = vt$. For $v > 0$, the long-wavelength components of the random h_j which destroy the order for $E < E_T$ will be cut off by time averaging.[7] This can be shown explicitly for $E \gg E_T$ for arbitrary pinning strength in any dimension.

A schematic phase diagram for $d < 4$ is shown in Fig. 1(b). In this case, the long-wavelength components of the randomness destroy the order at $E = 0$ for *all* D, and there will *always* be a threshold field. The regime with no threshold will hence not exist, i.e., $D_M = D_c = 0$. A CDW with random impurities in $d < 4$ will hence show true long-range order (i.e., δ-function Bragg peaks in the structure factor) *only* for $E > E_T$ when $v > 0$. The intensity of these peaks should scale as $(E - E_T)^{2\beta}$ where the exponent β will presumably depend on d.

The dependence of the threshold field on D for small D will be given[4,12] by $E_T \sim D^{4/4-d}$. This can be most easily derived by assuming that the renormalization-group fixed point which controls the threshold behavior occurs at D^* and E^* both of order 1, and by noting that for $E, D \ll 1$, the renormalized parameters at a length scale L are $E(L) \sim L^2 E$ and $D(L) \sim L^{(4-d)/2} D$. The Lee-Rice pinning length[4] L^* (for a system with weak pinning) is in fact the length scale at which the effective pinning strength becomes of order of the CDW stiffness and hence "strong," i.e., $D(L^*) \sim D^* \sim 1$. The critical behavior near threshold will thus be the same in $d < 4$ for both strong and weak pinning provided defects in the CDW can be ignored.

Finally, I make several comments concerning the experiments on $NbSe_3$.[1-3] Firstly, it appears that the impurities distort the CDW sufficiently little so that even if there is no true long-range order for $E = 0$, the CDW correlation length is larger than 1 μm and hence there are apparent Bragg peaks.[13] It is thus reasonable to assume that the impurities do not cause destruction of the local CDW order suggesting that defects do *not* play an important role. Secondly, the current-voltage data near threshold[1-3] can probably be relatively well fitted by a ζ between 1 and 2 with some rounding due to macroscopic inhomogeneities near E_T. Thus the mean-field value $\zeta = \frac{3}{2}$ appears to fit the experiments considerably better than previous calculations.

I have assumed throughout this paper that for $E > E_T$ the average phase moves uniformly. It can be shown that the uniformly moving mean-field solution is *stable* to small fluctuations. However, for any finite N_I, there will be large oscillations in $d\bar{\varphi}/dt$ near E_T, with frequency $2\pi/T$. The experiments on small samples of $NbSe_3$ show large current oscillations (narrow-band noise) which are the same magnitude as the dc current very near threshold. Several explanations for these oscillations have been advanced,[6,14,15] but only the idea of a macroscopic dynamical instability of the uniformly moving state[15] has really addressed the question of the existence of the oscillations in the limit of a large system. In light of the large natural length scales discussed above, it may well be possible to explain the oscillations as a finite-size effect with a diverging amplitude as $E \to E_T^+$ caused by the divergent correlation length, $\xi \sim (E - E_T)^{-\nu}$, in the moving state. Since the correlation length measures the size of regions which move coherently, the current oscillations will become of relative order 1 when ξ reaches the size of the sample.

We note that a phenomenon closely related to CDW pinning is the critical-current behavior of flux lattices in type-II superconductors weakly pinned by impurities. In these systems, large uniform velocity fluctuations of the flux lattices are *not* observed,[16] although the concomitant interference effects of an ac plus a dc force on the dc velocity are observed in both $NbSe_3$ (Refs. 7 and 8) and flux-lattice motion.[16] This difference may be due to the relatively larger samples. A detailed study of the flux-flow behavior of weakly pinned flux lattices near to the critical current would be very interesting.

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