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Geometric Implementation of Hypercubic Lattices with Noninteger Dimensionality by Use of Low Lacunarity Fractal Lattices

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It is claimed that the abstract analytic continuation of hypercubic lattices to noninteger dimensionalities can be implemented explicitly by certain fractal lattices of low lacunarity. These lattices are special examples of Sierpinski carpets. Their being of low lacunarity means that they are arbitrarily close to being translationally invariant. The claim is substantiated for the Ising model in $D = 1+\epsilon$ dimensions, and for resistor network models with $1 < D < 2$.

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Sets whose dimensionality d is not an integer enter statistical physics from two separate directions: continuous ϵ expansions near an integer d in the theory of critical phenomena,[1-3] and fractals.[4-6] The ϵ expansions involve formal analytic continuations of momentum integrals, e.g., $\int d^dq \to \int q^{d-1}dq$, or of recursion relations constructed for d-dimensional hypercubic lattices.[7,8] Such spaces have never been implemented, nevertheless it is postulated that they are *translationally invariant.* The general belief in *universality* is that for given symmetry of the order parameter and range of interaction, a system's critical properties depend *solely* on the dimensionality d.[9] In particular, all Ising models with short-range interactions and given $d \geq 1$ are believed to exhibit identical critical properties, with the critical temperature T_c decreasing to zero at the lower critical dimensionality $d_l = 1$.

Fractals, to the contrary, are fully specified geometric shapes.[4,5] The goal of the present paper is to show that, from the viewpoint of certain important problems, the formal fractional dimensional spaces with $1 < D < 2$ can be implemented by suitable fractals. These fractals may also be useful in performing explicitly some other calculations for such problems.

One very important critical exponent is $\nu \equiv 1/y$, which characterizes the divergence of the correlation length ξ near the critical temperature, through $\xi \sim (T - T_c)^{-\nu}$. Both the real-space Migdal hypercubic recursion relations[7,8] and the $(d-1)$-dimensional interface energy model[3] predict that in $d = 1+\epsilon$ dimensions the Ising model satisfies $y = \epsilon + O(\epsilon^2)$. This result is generally believed to be exact for small ϵ, but until now it could not be checked on any explicit $(1+\epsilon)$-dimensional geometrical shape. In this paper we propose to show that it is satisfied in the limit on a suitably constructed fractal lattice.[4,5] We also show that at $T \ll T_c$ the temperature on this fractal lattice scales with the exponent ϵ, as also found for the abstract hypercubic lattices. It has recently been shown[6] that the critical phenomena on a fractal lattice depend not only on its (noninteger) fractal dimensionality D, but also on other geometric and topological parameters. The parameter that will be needed here is *lacunarity* [Ref. 5(a), Chaps. 34 and 35], which measures the deviation of a fractal from being translationally invariant.[10] One measure of lacunarity is obtained by considering the mass of a fractal contained in a sphere (or cube) of radius (side) ρ. This mass can be written as $F\rho^D$, where the expression ρ^D simply interpolates the corresponding expression for lines, planes, etc., but the prefactor F is very different: It is *not* a numerical factor but a random variable. One basic measure of lacunarity is the mean-square deviation of F divided by its square mean.

The present paper proposes to show that at the limit of low lacunarity, the physical properties of these fractals (which are concrete and geometrically implementable geometric shapes) become identical to those of the abstract analytically continued hypercubic lattices.

Two examples are investigated. (a) We recover $y=\epsilon+O(\epsilon^2)$ for the Ising problem on $(1+\epsilon)$-dimensional fractals. (b) We find that the resistance of a D-dimensional resistor network $(1<D<2)$ scales as L^{2-D} with the linear size L. Thus, our low-lacunarity fractal lattices yield explicit geometric implementations of the systems with noninteger dimensionality that are postulated, e.g., in Refs. 1–3. We do *not* claim that these implementations are unique, nevertheless we think that our demonstration of their existence is of help in assessing the relevance of the abstract analytical calculations.

The fractals we consider are special Sierpinski carpets [Ref. 5a, Chap. 14]. Given two integers b and c, each square subdivides into b^2 intermediate squares, each of which subdivides further into c^2 small squares. Each stage of the fractal's construction cuts out "tremas," each of which is made of l^2 small squares in the center of each intermediate square. Thus, $b^2(c^2-l^2)$ small squares are left in, and the fractal dimensionality is given by[4-6]

$$D=\ln[b^2(c^2-l^2)]/\ln(bc). \tag{1}$$

The case $b=2$, $c=3$, $l=1$ is shown in Fig. 1. It can be shown[5b] that when b tends to infinity, lacunarity tends to zero. In order to achieve $D=1+\epsilon$, we let $c\to\infty$, we keep $c-l=q$ a finite constant, and we make sure that $\ln b \ll \ln c$. Thus

$$D\simeq 1+\ln(2qb)/\ln(bc),\quad c\to\infty. \tag{2a}$$

To achieve $D=2-\epsilon$, let $c\to\infty$, and keep l a finite constant. Thus,

$$D\simeq 2-l^2/c^2\ln(bc),\quad c\to\infty. \tag{2b}$$

Finally, any prescribed value of D with $1<D<2$ can be achieved by picking any exponent δ, with $0<\delta<D$, and setting

$$l^2=c^2-c^\delta,\quad b=c^{(D-\delta)/(2-D)},\quad c\to\infty. \tag{2c}$$

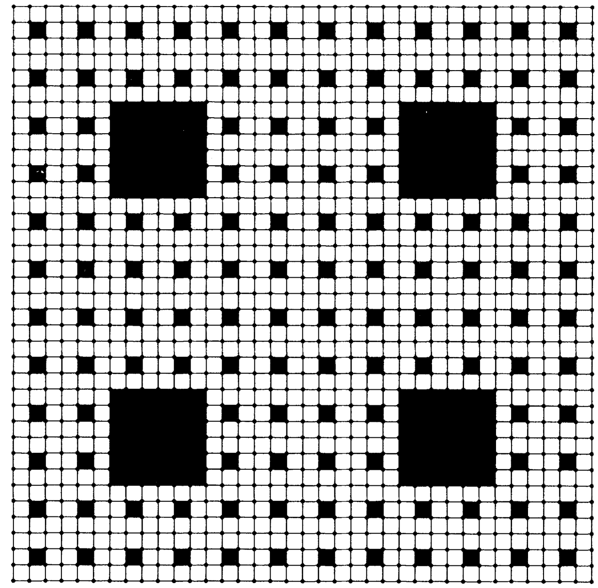

FIG. 1. The fractal Sierpinski lattice described in the text, with $b=2$, $c=3$, $l=1$, is shown here after the second construction stage. The points denote the lattice sites. Whenever a bond between nearest-neighbor sites lies on the boundary of a (black) cut out trema, it is characterized by K_w; otherwise, it is characterized by K.

The above procedure is iterated until nearest-neighbor lattice sites are located a distance a from each other. No lattice sites are placed in the *interior* of the cut out tremas. Every two nearest-neighbor sites are connected by a bond. We create a physical model, as in Ref. 6, by placing on each lattice site a spin-$\frac{1}{2}$ Ising moment with nearest-neighbor interactions. This model is investigated in the case (2a) with $b,c\to\infty$.

It is necessary to distinguish two sorts of nearest-neighbor bonds (in units of kT): those on the boundary of a cutout, K_w, and internal bonds, K. Our renormalization-group scheme moves all the bonds within a dedecorated square to its perimeter, and then decimates.[6-8] Although this procedure is not exact in general, we argue that the results are probably exact to order ϵ, because the exponent y is independent of b and c; that is, the recursion relations are the same after a single rescaling transformation of factor $(bc)^n$, and after n rescaling transformations of factor bc. The analytic continuation of the hypercubic lattices to $d=1+\epsilon$ is also trusted because the result $y=\epsilon+O(\epsilon^2)$ is independent of the rescaling factor.

The new coupling constants K' and K_w' are now found to be[6]

$$\tanh K' = \tanh^{bl}[b(c-l-1)K+2bK_w]\tanh^{b(c-l)}[bcK],$$
$$\tanh K_w' = \tanh^{bl}[\tfrac{1}{2}b(c-l-1)K+(b+1)K_w-\tfrac{1}{2}K]\tanh^{b(c-l)}[\tfrac{1}{2}(bc-1)K+K_w]. \quad (3)$$

From now on, we limit ourselves to the case $b,c \gg 1$, $c-l=q=\text{const}$. A schematic flow diagram using the coordinates $\tanh K$ and $\tanh K_w$ is shown in Fig. 2. We find five fixed points, A,B,C,D,E. Two of them, namely $C(K=0,K_w=0)$ and $E(\infty,\infty)$, are trivial stable fixed points, corresponding to the infinite and zero-temperature phases, respectively.

Linearizing the recursion relations near point E with the variables K and K_w, we find that $b(c-l)$ is an eigenvalue corresponding to the eigenvector $K/K_w=2$. This eigenvalue can be rewritten as $(bc)^{\tilde{y}}$, with $\tilde{y}=\epsilon+O(\epsilon^2)$, as is found by many low-temperature recursion relations on abstract translationally invariant systems.[3,11] This value of the exponent near the zero-temperature fixed point describes the scaling of the surface tension. The value $\tilde{y}=D-1$ is probably related to the fact that the perimeter of a (randomly chosen) domain of excited spins scales as L^{D-1} with the linear size L.

The point $D(0,\infty)$ describes the case with infinitely strong interactions on the boundaries of the "holes." It is stable in the $\tanh K_w$ direction and unstable in the $\tanh K$ direction. When $b,c \gg 1$, the coordinates of the point A are $(1/(bc)^{1+1/(qb-1)}, \infty)$. This point is unstable, with the eigenvalues $qb=(bc)^{\ln(qb)/\ln(bc)}$, and $qb=\exp[-\frac{1}{2}(bc)^{-1/(qb-1)}]$, corresponding to eigenvectors in the $\tanh K$ and $\tanh K_w$ directions, respectively. Notice that for lattices near one dimension [$D=1+\epsilon$, cf. Eq. (2a)] the first eigenvalue (the larger one) can be written as $(bc)^{\epsilon}$, so the related critical exponent is $y_A=\epsilon$, independent of bc.

The analysis of the fixed point B involves additional algebra. Using Eqs. (3), and taking $b,c \gg 1$, $\ln c \gg b$, we first find the location of B in the parameter space:

$$K_B=\ln\{b(c-q)/2[b(q-1)-1]\}+O(K_{wB},e^{-4K_B}/b),$$
$$K_{wB}=\exp\{-2[b(c-q)]^{1/2}\}[1+O(e^{-4K_B})].$$

The terms of the 2×2 matrix of the partial derivatives are given, to the leading order, by

$$\frac{\partial K'}{\partial K}\simeq b(q-1)[1-\tfrac{2}{3}e^{-4K_B}]; \quad \frac{\partial K'}{\partial K_w}\simeq 2b;$$
$$\frac{\partial K_w'}{\partial K}\simeq 2[b(q-1)-1][b(c-q)]^{1/2}\exp\{-2[b(c-q)]^{1/2}\}; \quad \frac{\partial K_w'}{\partial K_w}\simeq 4(b+1)[b(c-q)]^{1/2}\exp\{-2[b(c-q)]^{1/2}\}. \quad (4)$$

The eigenvalues are $\lambda_1\simeq b(q-1)[1-\frac{2}{3}e^{-4K_B}]$ and $\lambda_2=16(q-1)^{-1}[b(c-q)]^{1/2}\exp\{-2[b(c-q)]^{1/2}\}$. As $c\to\infty$, $\lambda_2\to0$, so that it represents an irrelevant field. Defining y via $\lambda_1=(bc)^y$, the limit $c\to\infty$ yields $y\simeq\ln[b(q-1)]/\ln(bc)$. For $b\gg1$, this simplifies further to $y\simeq\ln b/\ln(bc)$. On the other hand, Eq. (2a) yields $\epsilon=D-1\simeq\ln b/\ln(bc)$. Thus we have recovered the result of Midgal[7] and Kadanoff[8] and Wallace and Zia[3]: $y=\epsilon+O(\epsilon^2)$. We emphasize again that, this result being independent of b and of c, we believe it to be exact.

Next, using fractal lattices of the same family for any D satisfying $1<D<2$ [see Eqs. (2)] we study the problem of electric conduction on our lattices. In this case each bond on the boundary

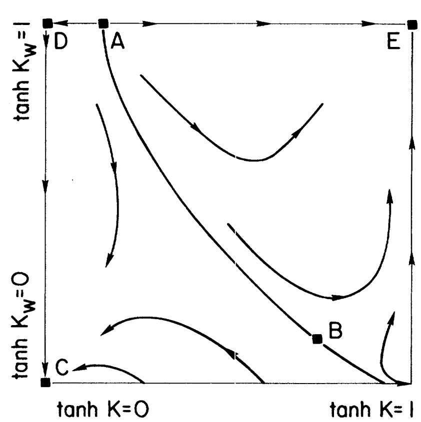

FIG. 2. A schematic flow diagram in the parameter space ($\tanh K, \tanh K_w$) for $l=2$ and $b,c\gg1$.

of the eliminated areas is taken to be a resistor with resistance R_w; similarly, what used to be a K bond is now taken to be a resistor R. The resistor-moving renormalization-group scheme yields the following equations:

$$R'=\frac{bl}{b(c-l-1)/R+2b/R_w}+\frac{b(c-1)}{bc/R},\quad R_w'=\frac{bl}{b(c-l-1)/2R+(b+1)/R_w-1/2R}+\frac{b(c-l)}{(bc-1)/2R+1/R_w}. \qquad (5)$$

Equations (5) may be written as a single recursion relation in the variable $\alpha=R_w/R$. When b, $c\to\infty$, one has $\alpha\to 2$. Substituting in Eqs. (5), we find that

$$R'=[l/(c-l)+(c-l)/c]R=(bc)^{\bar{\zeta}}R, \qquad (6)$$

where $\bar{\zeta}=2-D$. This result applies to all the cases considered in Eqs. (2). Again, this agrees with the result one expects for the abstract analytically continued, translationally invariant lattices.

Our several examples suggest that, within the Migdal approximation, our low-lacunarity fractals and the abstract "hypercubic" lattices have the same physical properties, for general noninteger D. It is clear, however, that the general statement near the beginning of this article requires further tests: One should compare general-D low- and high-temperature expansions, other renormalization-group schemes, exact calculations, etc. We hope that this paper will stimulate such further studies.

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Can Nuclear Interactions Be Long Ranged?

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A renormalizable relativistic quantum field theory of nuclear interactions is shown to possess not only Yukawa-type solutions, but also a topologically nontrivial one. It corresponds to a hadronic monopole, called a *hadroid*. Experimental evidence suggesting the existence of such a nuclear state is considered.

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The possibility that hadronic forces can be long ranged in the ground state was considered by Lee and Yang a number of years ago.[1] Their argument was based on the assumption that hadronic interactions are invariant under local non-Abelian gauge transformations. At that time it was believed that such invariance necessitated the existence of massless vector bosons, leading to a formal equivalence of the non-Abelian theory with electromagnetic gauge transformation. We now

