Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations

R. B. Laughlin
Lawrence Livermore National Laboratory, University of California, Livermore, California 94550

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This Letter presents variational ground-state and excited-state wave functions which describe the condensation of a two-dimensional electron gas into a new state of matter.

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The "$\frac{1}{3}$" effect, recently discovered by Tsui, Störmer, and Gossard,[1] results from the condensation of the two-dimensional electron gas in a GaAs-Ga_xAl_{1-x}As heterostructure into a new type of collective ground state. Important experimental facts are the following: (1) The electrons condense at a particular density, $\frac{1}{3}$ of a full Landau level. (2) They are capable of carrying electric current with little or no resistive loss and have a Hall conductance of $\frac{1}{3}e^2/h$. (3) Small deviations of the electron density do not affect either conductivity, but large ones do. (4) Condensation occurs at a temperature of ~ 1.0 K in a magnetic field of 150 kG. (5) The effect occurs in some samples but not in others. The purpose of this Letter is to report variational ground-state and excited-state wave functions that I feel are consistent with all the experimental facts and explain the effect. The ground state is a new state of matter, a quantum fluid the elementary excitations of which, the quasielectrons and quasiholes, are fractionally charged. I have verified the correctness of these wave functions for the case of small numbers of electrons, where direct numerical diagonalization of the many-body Hamiltonian is possible. I predict the existence of a sequence of these ground states, decreasing in density and terminating in a Wigner crystal.

Let us consider a two-dimensional electron gas in the x-y plane subjected to a magnetic field H_0 in the z direction. I adopt a symmetric gauge vector potential $\vec{A} = \frac{1}{2}H_0[x\hat{y}-y\hat{x}]$ and write the eigenstates of the ideal single-body Hamiltonian $H_{sp} = |(\hbar/i)\nabla - (e/c)\vec{A}|^2$ in the manner

$$|m,n\rangle = (2^{m+n+1}\pi m!\,n!)^{-1/2}\exp[\tfrac{1}{4}(x^2+y^2)]\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^m\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^n\exp[-\tfrac{1}{2}(x^2+y^2)]\,, \tag{1}$$

with the cyclotron energy $\hbar\omega_c = \hbar(eH_0/mc)$ and the magnetic length $a_0 = (\hbar/m\omega_c)^{1/2} = (\hbar c/eH_0)^{1/2}$ set to 1. We have

$$H_{sp}|m,n\rangle = (n+\tfrac{1}{2})|m,n\rangle. \tag{2}$$

The manifold of states with energy $n+\frac{1}{2}$ constitutes the nth Landau level. I abbreviate the states of the lowest Landau level as

$$|m\rangle = (2^{m+1}\pi m!)^{-1/2} z^m \exp(-\tfrac{1}{4}|z|^2)\,, \tag{3}$$

where $z = x+iy$. $|m\rangle$ is an eigenstate of angular momentum with eigenvalue m. The many-body Hamiltonian is

$$H = \sum_j\{|(\hbar/i)\nabla_j - (e/c)\vec{A}_j|^2 + V(z_j)\} + \sum_{j>k} e^2/|z_j - z_k|\,, \tag{4}$$

where j and k run over the N particles and V is a potential generated by a uniform neutralizing background.

I showed in a previous paper[2] that the $\frac{1}{3}$ effect could be understood in terms of the states in the lowest Landau level solely. With $e^2/a_0 \lesssim \hbar\omega_c$, the situation in the experiment, quantization of interelectronic spacing follows from quantization of angular momentum: The only wave functions composed of states in the lowest Landau level which describe orbiting with angular momentum m about the center of mass are of the form

$$\psi = (z_1 - z_2)^m (z_1 + z_2)^n \exp[-\tfrac{1}{4}(|z_1|^2 + |z_2|^2)]. \tag{5}$$

My present theory generalizes this observation to N particles.

I write the ground state as a product of Jastrow functions in the manner

$$\psi = \{\prod_{j<k} f(z_j - z_k)\}\exp(-\tfrac{1}{4}\textstyle\sum_l |z_l|^2)\,, \tag{6}$$

and minimize the energy with respect to f. We observe that the condition that the electrons lie in the lowest Landau level is that $f(z)$ be polynomial in z. The antisymmetry of ψ requires that f be odd. Conservation of angular momentum requires that $\prod_{j<k} f(z_j - z_k)$ be a homogeneous polynomial of degree M, where M is the total angular momentum. We have, therefore, $f(z) = z^m$, with m odd. To determine which m minimizes the energy, I write

$$|\psi_m|^2 = |\{\prod_{j<k}(z_j - z_k)^m\}\exp(-\tfrac{1}{4}\sum_l |z_l|^2)|^2 = e^{-\beta\Phi}, \tag{7}$$

where $\beta = 1/m$ and Φ is a classical potential energy given by

$$\Phi = -\sum_{j<k} 2m^2 \ln|z_j - z_k| + \tfrac{1}{2} m \sum_l |z_l|^2. \tag{8}$$

Φ describes a system of N identical particles of charge $Q = m$, interacting via logarithmic potentials and embedded in a uniform neutralizing background of charge density $\sigma = (2\pi a_0^2)^{-1}$. This is the classical one-component plasma (OCP), a system which has been studied in great detail. Monte Carlo calculations[3] have indicated that the OCP is a hexagonal crystal when the dimensionless plasma parameter $\Gamma = 2\beta Q^2 = 2m$ is greater than 140 and a fluid otherwise. $|\psi_m|^2$ describes a system uniformly expanded to a density of $\sigma_m = m^{-1}(2\pi a_0^2)^{-1}$. It minimizes the energy when σ_m equals the charge density generating V.

In Table I, I list the projection of ψ_m for three particles onto the lowest-energy eigenstate of angular momentum $3m$ calculated numerically. These are all nearly 1. This supports my assertion that a wave function of the form of Eq. (6) has adequate variational freedom. I have done a similar calculation for four particles with Coulombic repulsions and find projections of 0.979 and 0.947 for the $m = 3$ and $m = 5$ states.

ψ_m has a total energy per particle which for small m is more negative than that of a charge-density wave (CDW).[4] It is given in terms of the radial distribution function $g(r)$ of the OCP by

$$U_{\text{tot}} = \pi \int_0^\infty \frac{e^2}{r}[g(r) - 1] r dr. \tag{9}$$

In the limit of large Γ, U_{tot} is approximated within a few percent by the ion disk energy:

$$U_{\text{tot}} \simeq -\sigma_m \int \frac{e^2}{|r|} d^2r + \frac{\sigma_m^2}{2} \iint \frac{e^2}{|r_{12}|} dr_1^2 dr_2^2 = (4/3\pi - 1) 2e^2/R, \tag{10}$$

where the integration domain is a disk of radius $R = (\pi\sigma_m)^{-1/2}$. At $\Gamma = 2$ we have the exact result[5] that $g(r) = 1 - \exp[-(r/R)^2]$, giving $U_{\text{tot}} = -\frac{1}{2}\pi^{1/2} e^2/R$. At $m = 3$ and $m = 5$ I have reproduced the Monte Carlo $g(r)$ of Caillol *et al.*[3] using the modified hypernetted chain technique described by them. I obtain $U_{\text{tot}} = (-0.4156 \pm 0.0012)e^2/a_0$ and $U_{\text{tot}}(5) = (-0.3340 \pm 0.0028)e^2/a_0$. The corresponding values for the charge-density wave[4] are $-0.389 e^2/a_0$ and $-0.322 e^2/a_0$. U_{tot} is a smooth function of Γ. I interpolate it crudely in the manner

$$U_{\text{tot}}(m) \simeq \frac{0.814}{\sqrt{m}} \left(\frac{0.230}{m^{0.64}} - 1 \right) \frac{e^2}{a_0}. \tag{11}$$

This interpolation converges to the CDW energy near $m = 10$. The actual crystallization point cannot be determined from that of the OCP since the CDW has a lower energy than the crystal described by ψ_m for $m > 71$.

I generate the elementary excitations of ψ_m by piercing the fluid at z_0 with an infinitely thin solenoid and passing through it a flux quantum $\Delta\varphi = hc/e$ adiabatically. The effect of this operation on the single-body wave functions is

$$(z - z_0)^m \exp(-\tfrac{1}{4}|z|^2) \to (z - z_0)^{m+1} \exp(-\tfrac{1}{4}|z|^2). \tag{12}$$

Let us take as approximate representations of these excited states

$$\psi_m^{+z_0} = A_{z_0}\psi_m = \exp(-\tfrac{1}{4}\sum_l |z_l|^2)\{\prod_i (z_i - z_0)\}\{\prod_{j<k}(z_j - z_k)^m\}, \tag{13}$$

and

$$\psi_m^{-z_0} = A_{z_0}^+ \psi_m = \exp(-\tfrac{1}{4}\sum_l |z_l|^2)\left\{\prod_i \left(\frac{\partial}{\partial z_i} - \frac{z_0}{a_0^2}\right)\right\}\{\prod_{j<k}(z_j - z_k)^m\}, \tag{14}$$

TABLE I. Projection of variational three-body wave functions ψ_m in the manner $\langle\psi_m|\Phi_m\rangle/(\langle\psi_m|\psi_m\rangle\langle\Phi_m|\Phi_m\rangle)^{1/2}$. Φ_m is the lowest-energy eigenstate of angular momentum $3m$ calculated with $V = 0$ and an interelectronic potential of either $1/r$, $-\ln(r)$, or $\exp(-r^2/2)$.

m	$1/r$	$-\ln(r)$	$\exp(-r^2/2)$
1	1	1	1
3	0.999 46	0.996 73	0.999 66
5	0.994 68	0.991 95	0.999 39
7	0.994 76	0.992 95	0.999 81
9	0.995 73	0.994 37	0.999 99
11	0.996 52	0.995 42	0.999 96
13	0.997 08	0.996 15	0.999 85

for the quasihole and quasielectron, respectively. For four particles, I have projected these wave functions onto the analogous ones computed numerically. I obtain 0.998 for ψ_3^{-0} and 0.994 for ψ_5^{-0}. I obtain 0.982 for $\overline{\psi}_3^{\mp\bar{0}} = \{\prod_i (z_i - \bar{z})\}\psi_3$, which is ψ_3^{+0} with the center-of-mass motion removed.

These excitations are particles of charge $1/m$. To see this let us write $|\psi^{+z_0}|^2$ as $e^{-\beta\Phi'}$, with $\beta = 1/m$ and

$$\Phi' = \Phi - 2\sum_l \ln|z_l - z_0|. \tag{15}$$

Φ' describes an OCP interacting with a phantom point charge at z_0. The plasma will completely screen this phantom by accumulating an equal and opposite charge near z_0. However, since the plasma in reality consists of particles of charge 1 rather than charge m, the real accumulated charge is $1/m$. Similar reasoning applies to ψ^{-z_0} if we approximate it as $\prod_j (z_j - z_0)^{-1} P_{z_0}\psi_3$, where P_{z_0} is a projection operator removing all configurations in which any electron is in the single-body state $(z - z_0)^0 \exp(-\frac{1}{4}|z|^2)$. The projection of this approximate wave function onto $\psi_3^{-z_0}$ for four particles is 0.922. More generally, one observes that far away from the solenoid, adiabatic addition of $\Delta\varphi$ moves the fluid rigidly by exactly one state, per Eq. (12). The charge of the particles is thus $1/m$ by the Schrieffer counting argument.[6]

The size of these particles is the distance over which the OCP screens. Were the plasma weakly coupled ($\Gamma \leq 2$) this would be the Debye length $\lambda_D = a_0/\sqrt{2}$. For the strongly coupled plasma, a better estimate is the ion-disk radius associated with a charge of $1/m$: $R = \sqrt{2}\, a_0$. From the size we can estimate the energy required to make a particle. The charge accumulated around the phantom in the Debye-Hückel approximation is

$$\delta\rho = \frac{e/m}{2\pi\lambda_D^2} K_0(r/\lambda_D),$$

where K_0 is a modified Bessel function of the second kind. The energy required to accumulate it is

$$\Delta_{\text{Debye}} = \frac{1}{2}\iint \frac{\delta\rho\,\delta\rho}{|r_{12}|} = \frac{\pi}{4\sqrt{2}} \frac{1}{m^2} \frac{e^2}{a_0}. \tag{16}$$

This estimate is an upper bound, since the plasma is strongly coupled. To make a better estimate let $\delta\rho = \sigma_m$ inside the ion disk and zero outside, to obtain

$$\Delta_{\text{disk}} = \frac{3}{2\sqrt{2}\pi} \frac{1}{m^2} \frac{e^2}{a_0}. \tag{17}$$

For $m = 3$, these estimates are $0.062e^2/a_0$ and $0.038e^2/a_0$. This compares well with the value $0.033e^2/a_0$ estimated from the numerical four-particle solution in the manner

$$\Delta \simeq \frac{1}{2}\{E(\psi_3^{-0}) + E(\overline{\psi}_3^{\mp\bar{0}}) - 2E(\psi_3)\}, \tag{18}$$

where $E(\psi_3)$ denotes the eigenvalue of the numerical analog of ψ_3. This expression averages the electron and hole creation energies while subtracting off the error due to the absence of V. I have performed two-component hypernetted chain calculations for the energies of $\psi_3^{+z_0}$ and $\psi_3^{-z_0}$. I obtain $(0.022 \pm 0.002)e^2/a_0$ and $(0.025 \pm 0.005)e^2/a_0$. If we assume a value $\epsilon = 13$ for the dielectric constant of GaAs, we obtain $0.02e^2/\epsilon a_0 \simeq 4$ K when $H_0 = 150$ kG.

The energy to make a particle does not depend on z_0, so long as its distance from the boundary is greater than its size. Thus, as in the single-particle problem, the states are degenerate and there is no kinetic energy. We can expand the creation operator as a power series in z_0:

$$A_{z_0} = \sum_{j=0}^{N} A_j(z_1 \cdots, z_N) z_0^{N-j}. \tag{19}$$

These A_j are the elementary symmetric polynomials,[7] the algebra of which is known to span the set of symmetric functions. Since every antisymmetric function can be written as a symmetric function times ψ_1, these operators and their adjoints generate the entire state space. It is thus appropriate to consider them N linearly independent particle creation operators.

The state described by ψ_m is incompressible because compressing or expanding it is tantamount to injecting particles. If the area of the system is reduced or increased by δA the energy rises by $\delta U = \sigma_m \Delta |\delta A|$. Were this an elastic solid characterized by a bulk modulus B, we would have $\delta U = \frac{1}{2}B(\delta A)^2/A$. Incompressibility causes the longitudinal collective excitation roughly equivalent to a compressional sound wave to be absent, or more precisely, to have an energy $\sim\Delta$ in the long-wavelength limit. This facilitates current conduction with no resistive loss at zero temperature. Our prototype for this behavior is full Landau level ($m = 1$) for which this collective excitation occurs at $\hbar\omega_c$. The response of this system to compressive stresses is analogous to the response of a type-II superconductor to the application of a magnetic field. The system first generates Hall currents without compressing, and then at a critical stress collapses by an area quantum $m 2\pi a_0^2$

and nucleates a particle. This, like a flux line, is surrounded by a vortex of Hall current rotating in a sense opposite to that induced by the stress.

The role of sample impurities and inhomogeneities in this theory is the same as that in my theory of the ordinary quantum Hall effect.[8] The electron and hole bands, separated in the impurity-free case by a gap 2Δ, are broadened into a continuum consisting of two bands of extended states separated by a band of localized ones. Small variations of the electron density move the Fermi level within this localized state band as the extra quasiparticles become trapped at impurity sites. The Hall conductance is $(1/m) \times (e^2/h)$ because it is related by gauge invariance to the charge of the quasiparticles e^* by $\sigma_{\text{Hall}} = e^*e/h$, whenever the Fermi level lies in a localized state band. As in the ordinary quantum Hall effect, disorder sufficient to localize all the states destroys the effect. This occurs when the collision time τ in the sample in the absence of a magnetic field becomes smaller than $\tau < \hbar/\Delta$.

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