Comments on "Transverse Electromagnetic Waves with $\vec{E} \parallel \vec{B}$"

The claim by Chu and Ohkawa[1] that a general class of transverse electromagnetic (EM) waves with parallel $\vec{E}$ and $\vec{B}$ fields exists is shown globally to be false by using symmetry and topological arguments.

For simplicity, let us consider the symmetry arguments in vacuum. $\nabla\times\vec{B}=k\vec{B}$ is not invariant under space inversion, yet the remaining three equations of Maxwell's equations are invariant.[2] In other words, in R^3 the remaining three Maxwell's equations require that $\vec{E}$ be a vector and $\vec{B}$ be a pseudovector. But $\vec{B}$ being a pseudovector is inconsistent with $\nabla\times\vec{B}=k\vec{B}$ except for $\vec{B}=\vec{0}$, where k is a scalar. In fact, if we replace Ampere's law by $\nabla\times\vec{B}=k\vec{B}$, the field variables of this new set of equations cannot form an EM field tensor as do F^{ij} of Maxwell's equations nor do they have the same properties. Thus, there are no transverse EM waves with $\vec{E}\parallel\vec{B}$ in R^3 in the global sense.

As an example consider the formation of standing waves in a cavity with mirrors, as suggested by Chu and Ohkawa. A left-handed circularly polarized EM wave in a vacuum can be described by

$$\vec{E}_l = kA[\hat{i}\cos(kz-\omega t)-\hat{j}\sin(kz-\omega t)],$$

$$\vec{B}_l = kA[\hat{i}\sin(kz-\omega t)+\hat{j}\cos(kz-\omega t)].$$

Now $\vec{E}_l$ and $\vec{B}_l$ satisfy $\nabla\times\vec{B}_l=(1/c)\partial\vec{E}_l/\partial t=k\vec{B}_l$, $\nabla\times\vec{E}_l=(-1/c)\partial\vec{B}_l/\partial t=k\vec{E}_l$, and $\nabla\cdot\vec{E}_l=\nabla\cdot\vec{B}=0$. The reflected wave is a right-handed polarized wave in the opposite direction:

$$\vec{E}_r = kA[\hat{i}\cos(kz-\omega t)+\hat{j}\sin(kz-\omega t)],$$

$$\vec{B}_r = kA[-\hat{i}\sin(kz-\omega t)+\hat{j}\cos(kz-\omega t)].$$

The resultant EM fields are $\vec{E}_t=2kA\hat{i}\cos(kz-\omega t)$, $\vec{B}_t=2kA\hat{j}\cos(kz-\omega t)$. This is a linearly polarized wave with $\vec{E}_t$ perpendicular to $\vec{B}_t$, and when $kz-\omega t=(\text{odd integer})\pi/2$ both $\vec{E}_t$ and $\vec{B}_t$ become zero vectors. Certainly they are "parallel" to each other, but they are so only at isolated points in space and time. The existence of EM waves with $\vec{E}\parallel\vec{B}$ is not excluded locally, but the global existence of the said waves in R^3 is explicitly excluded. This can also be shown by topological arguments in a simple closed cavity or in a simple torus.[3]

This example contained trivial boundary conditions. It may be possible to find examples in which more complex, nontrivial boundary conditions may lead to standing waves with $\vec{E}\parallel\vec{B}$ in the whole cavity. Evtuhov and Siegman[4] have proposed an example with nontrivial boundary conditions in which two $\lambda/4$ plates are introduced between the rod and the end mirrors of a laser cavity. A right-handed polarized EM wave traveling to the right is reflected with the same polarization by the end mirror producing circularly polarized standing waves with uniform energy density in the cavity. An analysis similar to that in the previous example yields the same conclusion, i.e., EM waves with $\vec{E}\parallel\vec{B}$ do not exist globally in the cavity proposed by Evtuhov and Siegman.

In summary, I have shown that there exist no global $\vec{E}\parallel\vec{B}$ waves in space with trivial topology (boundary conditions). The possible existence of such waves may require that the boundary conditions for the physical space be more complex and nontrivial.

The author would like to express his appreciation to J. Delettrez and R. L. McCrory for their suggestions for improvements.

K. K. Lee
Laboratory for Laser Energetics
University of Rochester
Rochester, New York 14623

Received 19 April 1982
PACS numbers: 03.50.De, 41.10.Hv, 52.35.Hr

[1] C. Chu and T. Ohkawa, Phys. Rev. Lett. 48, 837 (1982).

[2] J. D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1975).

[3] Assume that the space-time M is an orientable, connected, four-dimensional C^∞ manifold endowed with a Lorentz metric and $M=S\times R^1$. The three-manifold S is assumed to be compact without boundary. Then the source-free Maxwell's equations can be represented by $df=0$ and $\delta f=0$, where d is the exterior derivative, f is the EM field two-form, and $\delta\equiv d^*$, with the asterisk denoting the star operator. Thus f is a harmonic two-form. From de Rham's theorem, the existence of a nonzero harmonic two-form in S implies the nonvanishing of the second Betti number of S $[b_2(S)]$. In fact, from the Poincaré duality theorem, we have $b_1(S)=b_2(S)\neq 0$. Since $b_2(S)\neq 0$, then, although $\nabla\cdot\vec{B}=0$, there does not exist a vector potential $\vec{A}$ such that $\vec{B}=\nabla\times\vec{A}$ over the entire S. Consequently, no vector potential $\vec{A}$ satisfies Eq. (7) of Chu and Ohkawa globally in S and therefore transverse EM waves with $\vec{E}\parallel\vec{B}$ cannot exist globally in S. It should be noticed that the conclusion of Chu and Ohkawa was based on the local existence of a vector potential which satisfies their Eq. (7).

[4] V. Evtuhov and A. E. Siegman, Appl. Opt. 4, 142 (1965).

