Chaos in the Mixmaster Universe

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A four-dimensional system of nonlinear difference equations describing the evolution of the general relativistic "mixmaster" cosmological model is studied. The evolution splits into two parts: one given by a (chaotic) generalized Baker's transformation and the other by a (systematic) change of scales. The chaotic degrees of freedom are studied in detail: Their evolution may be described by a two-sided shift and the invariant measure for these variables is found. Strong ergodic properties (mixing, nonzero entropy) are deduced.

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The "mixmaster universe" is a homogeneous, anisotropic, Bianchi type -IX cosmological solution of the Einstein equations.[1] The fascinating properties of this vacuum space-time have motivated a host of studies in gravitational theory. Misner studied it as a possible explanation for the isotropy of the observed universe since particle horizons are removed during certain phases of its early evolution.[2] Belinskii, Khalatnikov, and Lifshitz suggested that the mixmaster system may be typical of the most general sort of behavior of Einstein's equations near a singularity.[3] The evolution of the mixmaster universe is entirely non-Newtonian since no Ricci, only Weyl, curvature is present. Here we will give a complete description of the mixmaster system in its oscillatory phase, close to a cosmological singularity, prove that the system has very strong ergodic properties, and demonstrate that the asymptotic evolution (joint probability distribution for all degrees of freedom) is exactly calculable in closed form.

The space-time metric has the form[4]

$$ds^2 = dt^2 - \sum_{i,j}\gamma_{ij}(t)\,\sigma^i(x)\sigma^j(x), \quad (1)$$

where $\sigma(x)$ are the differential forms for the type-IX homogeneous space satisfying $d\sigma^i = \epsilon^i{}_{jk}\sigma^j\Lambda\sigma^k$, $\epsilon^i{}_{jk}$ completely antisymmetric, and

$$\gamma_{ij} = \mathrm{diag}(e^{2\alpha}, e^{2\beta}, e^{2\gamma}). \quad (2)$$

The Einstein equations lead to three second-order ordinary differential equations for α, β, and γ (the axis scales) and one integration constraint. It is most convenient to use $\tau = \ln t$; as $\tau \to -\infty$ the initial singularity is approached. The qualitative features of the evolution have been well studied: The mixmaster system asymptotically approximates a sequence of different Kasner solutions.[5,6] During a typical Kasner cycle, one pair of axes oscillates out of phase and the third decreases. We define a *cycle* as the series of small oscillations during which there is the monotonic decrease of one of the expansion scales, say γ. At the end of a cycle, the roles of α, β, and γ are permuted, and in the next cycle one of α and β decreases monotonically while γ and the third axis oscillate. On approach to the singularity, the general pattern of oscillations (in cycles) and permutations repeats itself *ad infinitum*.

Elsewhere, it was noted that the evolution of one of the degrees of freedom (the number of oscillations in a cycle) is formally chaotic: It possesses positive metric and topological entropy, and is isomorphic to a Bernoulli shift.[7] Here we formulate the evolution for all five degrees of freedom of the mixmaster system and relate it to previous work. For two of the variables the evolution is essentially chaotic and we explicitly construct the invariant measure and a shift for these. The remaining variables undergo a systematic evolution which we also can calculate. Together, these results constitute a complete description of the mixmaster universe's evolution as a recurrent series of Kasner-like states.

To describe the evolution simply, we will define a set of four variables $\underline{r} = (u, x, s, \Sigma)$ on a surface of section $\alpha = 0$, $d\alpha/d\tau > 0$ by

$$\alpha = 0; \quad \frac{d\alpha}{d\tau} = sp_2(u) = \frac{s(1+u)}{(1+u+u^2)};$$

$$\beta = \frac{\Sigma x}{(x+1+u)}; \quad \frac{d\beta}{d\tau} = sp_1(u) = -\frac{su}{(1+u+u^2)}; \quad (3)$$

$$\gamma = \frac{\Sigma(1+u)}{(x+1+u)}; \quad \frac{d\gamma}{d\tau} = sp_3(u) = \frac{s(u+u^2)}{(1+u+u^2)};$$

where p_1, p_2, and p_3 are the Kasner type indices.[6]

τ equals Σ to within an additive constant. Physically, Σ and s are overall scale factors, while x and u determine the relative sizes of the axes and the axis velocities. The approximation used to calculate a discrete map on the surface of section is that the magnitude of the derivatives is large compared to the exponentials of α, β, and γ.

The basic map, $m:(u,x,s,\Sigma)\to(u',x',s',\Sigma')$ for $\tau'<\tau$, which we will discuss in more detail elsewhere,[13] is

$$(u',x')=\begin{cases}(u-1,x/(1+x)) \text{ if } \infty>u>1 \text{ (oscillations)},\\ (1/u-1,1+1/x) \text{ if } 1>u>0 \text{ (bounce)};\end{cases} \tag{4}$$

$$(s',\Sigma')=(s(1-u+u^2)/(1+u+u^2),\ \Sigma[1+(1+u+u^2)x/u(x+1+u)]),$$

where the variables range $\infty>u>0$, $\infty>x>0$, $\Sigma<0$, $s>0$. If we start from an initial state $(u_0, x_0, s_0, \Sigma_0)$, then $k_0=[u_0]$ (integer part of u_0) iterations of the $u>1$ map (oscillatory map) follow. The last step of a complete cycle is a "bounce" to a new set of initial conditions. Equivalent versions of (4) were derived and studied in Refs. 2, 3, and 5; the virtue of these variables over previous treatments is that each lies along a separate eigendirection for expansion or contraction of the mapping. For example, the map for Σ (s) is multiplication by a function of u and x which is always greater than (less than) 1; these scale parameters undergo a systematic increase (decrease) in magnitude.

Here, we are primarily interested in the induced map, $M:(u_0,x_0,s_0,\Sigma_0)\to(u_1,x_1,s_1,\Sigma_1)$, from the initial conditions of one bounce to those of the succeeding bounce (as $\tau\to-\infty$, so $\tau_1<\tau_0$):

$$\begin{aligned}(u_1,x_1)&=(1/(u_0-[u_0])-1,\ 1+[u_0]+1/x_0),\\ s_1&=s_0\frac{1-(u_0-[u_0])+(u_0-[u_0])^2}{1+u_0+u_0^2}=f_1(u_0),\\ \Sigma_1&=\Sigma_0\frac{1+[u_0]+1/x_0+1/(u_0-[u_0])}{1/x_0+1/(u_0+1)}=f_2(u_0,x_0).\end{aligned} \tag{5}$$

The inverse map, $M^{-1}:(u_0,x_0,s_0,\Sigma_0)\to(u_{-1},x_{-1},s_{-1},\Sigma_{-1})$, which describes the evolution away from the singularity, is constructed by inverting M above, with due attention to the ranges of the variables: $\infty>u_0>0$, $\infty>x_0>1$. These maps are approximate when the absolute value of s is larger than the exponential of Σ.

We restrict our attention now to the bounce map for the variables u and x. As $\tau\to-\infty$, expansion takes place along the u coordinate and contraction along the x coordinate. Figure 1 illustrates the action of the map in (x,u) space. The vertical region No. 1: $(\infty>x>1, 3>u>2)$ is squashed in the x direction and stretched in the u direction to map into region No. 2: $(4>x>3, \infty>u>0)$. Likewise, region No. 3 is mapped into region No. 4. For the entire plane, every vertical strip is deformed similarly and mapped into a corresponding horizontal strip. The map is a simple generalization of the well-studied Baker's transformation which is given as $(w',y')=(2w-[2w], (y+[2w])/2)$ for $w, y\in(0,1)$.[8] An essential feature which these maps share is dense periodic orbits; *there are no integrals of the motion under these circumstances.* This is strong evidence that the exact equations of motion have very complex dynamics.

The Baker's transformation is isomorphic to a Bernoulli shift with an alphabet of (0, 1). We now construct a (generalized) two-sided shift for the mixmaster map and show that it has a direct physical interpretation. Belinskii, Khalatnikov, and Lifshitz, without the above considerations, showed that the continued-fraction expansion (cfe) of u_0+1 of the initial state gives the sequence of integers $[u_i]+1$ generated by iteration of the map.[5] Let the cfe for a positive real number, y, be expressed by the semi-infinite se-

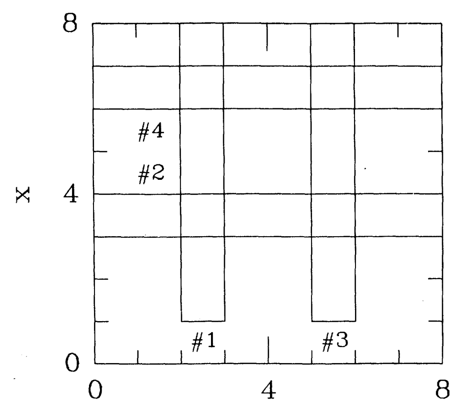

FIG. 1. Under an iteration of the map the vertical region No. 1 is compressed in the x direction, stretched in the u direction and mapped into horizontal region No. 2. Region No. 3, deformed similarily, maps into region No. 4. Each vertical strip in the plane undergoes a similar deformation and maps into a horizontal strip.

quence of integers i_k:

$$y=(i_0,i_1,i_2,\ldots)=i_0+1/[i_1+(1/i_2+\ldots)]. \quad (6)$$

Define $z_k=u_k+1$ and let z_0 have the expansion

$$z_0=(m_0,m_1,m_2,\ldots). \quad (7)$$

Then z_k (the kth iterate of z_0 under the map) is given as a k shift on the expansion for z_0; that is,

$$z_k=(m_k,m_{k+1},\ldots). \quad (8)$$

The expansion for any rational terminates in a finite number of steps. However, in a measure-theoretic sense, the case of physical interest is the expansion which continues indefinitely.[9] A given u_0 codes an infinite sequence of oscillations with $[u_i]$ oscillations in the ith cycle.

We find that the map x_k may be formulated similarily: Let x_0 be given by the cfe

$$x_0=(n_0,n_1,n_2,\ldots). \quad (9)$$

Then the kth iterate of x_0 is

$$x_k=(m_{k-1},m_{k-2},\ldots,m_0,n_0,n_1,n_2,\ldots), \quad (10)$$

where m_i are the elements of the expansion in z_0 above. The map for x and u is equivalent to a single shift on a two-sided infinite sequence of positive integers formed by concatenating the semi-infinite sequences for x_0 and z_0 to give $(\ldots m_{-2},m_{-1},m_0,n_0,n_1,n_2,\ldots)$. The inverse map, M^{-1}, is a shift in the opposite direction on the same two-sided sequence. For evolution as $\tau\to\infty$ (*away* from the singularity) the number of oscillations in the ith cycle is given by $[x_i]-1$. The doubly infinite sequence of integers thus corresponds to the entire past and future history (number of oscillations/cycle) of the mixmaster universe in its oscillatory phase. This symmetry between x and u [which is present in both the m and M maps, Eqs. (4) and (5)] is a direct consequence of the invertibility in time of the Einstein equations.

This shift is ergodic and strongly mixing and has well-defined metric and topological entropy.[10] A special class of maps are the axiom-A (AA) systems, whose essential (chaotic) features are preserved under small perturbations of the map.[11] This stability is a desirable feature for any model or approximate analysis of a real physical system. Two of the neccessary requirements for AA are uniform hyperbolicity on a compact phase space and separate, continuous, expanding and contracting dimensions. On the one hand, these assumptions may be overly restrictive for physical systems; on the other hand, AA systems are the only systems for which a complete, mathematically rigorous theory exists. Although the chaotic part of the mixmaster map fails to satisfy AA, it is interesting that it does incorporate several of the essential features of these systems: The map has (fixed) eigendirections for expansion $(\hat{u})$ and contraction $(\hat{x})$ and is hyperbolic $(du_1/du_0>1>dx_1/dx_0)$. However, its derivatives are not bounded away from 1, nor is phase space compact. As a consequence, orbits are *not* structurally stable: Small perturbations in the map can lead to attracting points. It seems to be an open question as to whether the exact mixmaster solutions would satisfy the requirements of AA on some compact subset of phase space.

The strong ergodic properties of the map for x and u prove that the system, while rigorously deterministic, is effectively stochastic; there is no possibility of forecasting the precise evolution of a mixmaster universe numerically, since any uncertainty in the initial conditions would quickly grow (exponentially in time) as large as the allowable phase space. In such a situation, the asymptotic probability distribution is of central interest, since it determines the value of all time-averaged observables.

For a mapping $\underline{r}_{n+1}=T(\underline{r}_n)$, where $\underline{r}\in R^n$, an invariant measure $\mu(\underline{r})$ is a solution of the functional equation

$$\mu(A)=\mu(T^{-1}A), \quad (11)$$

where A is a measurable set of R^n. If a unique absolutely continuous (with respect to Lebesgue measure), integrable μ exists, it gives the stationary probability distribution as $n\to\infty$ for almost every set of initial data. For the bounce map (and its inverse), we can easily verify that

$$\mu(u,x)=1/[(\ln 2)(1+ux)^2]. \quad (12)$$

This invariant measure corresponds to a probability for an integer k in the cfe representation of $\ln[(k+1)^2/(k^2+2k)]/\ln 2$.

If we define the ratio $\ln(s_1/s_0)=t_1$ and $\ln(\Sigma_1/\Sigma_0)=v_1$, then the probability distribution is

$$P(u,x,t,v)=\mu(u,x)\delta(t-\ln[f_1(u)])\,\delta(v-\ln[f_2(u,x)]), \quad (13)$$

where logarithms have been introduced to produce normalizable probability distributions. The expectation value of functions of these variables in the mixmaster universe follows by integration over the distribution.[12] Elsewhere we discuss the calculations of quantities of interest in the nonlinear dynamics of this model: the rate of divergence of trajectories, the decay of correlation functions, and the metric entropy.[13] We address the following points of physical interest: the rate of particle production by the curvature, the probability of mixing and removal of particle horizons, together with the decay of anisotropy.

Our analytic results will primarily be of interest to relativists, but the system we have studied may be of considerable interest to dynamicists as well because it is exactly soluble. We emphasize that the Einstein equations of motion lead directly to the map we have studied. The description by means of the generalized Baker's transformation suggests that the general solutions to the Einstein equations may have extraordinarily complex behavior. One is led to consider the questions: Under what circumstances will cosmological solutions to Einstein's equations display chaotic behavior and what variety may be expected? Of the Bianchi models, only the type-VIII and -IX appear to have a chaotic evolution. General relativity is an inherently nonlinear theory; to what extent is the mixmaster system representative of the nonlinear effects unique to general relativity?

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