Comment on "Hopping Mechanism Generating $1/f$ Noise in Nonlinear Systems"

In a recent Letter Arecchi and Lisi[1] claim to have observed $1/f$ noise associated with the chaotic dynamics of a certain driven nonlinear system. In this Comment we would like to point out that the essential features of the noise spectra observed by those authors, which are also observed in the case of the forced pendulum,[2-4] can be understood in terms of a simple model that is in good qualitative agreement with their results but which does not support their contention that such chaotic systems exhibit an intrinsic $1/f$ noise.

As pointed out by the authors of Ref. 1, and noted by others as well,[2-4] the high level of noise at low frequencies observed in chaotic nonlinear systems such as in Ref. 1 is related to a random hopping between two unstable strange attractors. This suggests that the overall signal voltage $V(t)$ can be modeled as the sum of signals associated with each attractor, $V_1(t)$ and $V_2(t)$, multiplied, respectively, by a random telegraph signal $T(t)$ and its complement $\bar{T}(t)$. Making the simplifying assumption (as suggested by the apparent random nature of T) that $V_1(t)$, $V_2(t)$, and $T(t)$ are statistically independent, we can write the autocorrelation function $R(\tau)$ of $V(t)$ as $R(\tau) = R_T(\tau)R_1(\tau) + R_{\bar{T}}(\tau)R_2(\tau)$ where R_1, R_2, R_T, and $R_{\bar{T}}$ are the autocorrelation functions of V_1, V_2, T, and $\bar{T}$. As is well known,[5] if τ_1 and τ_2 are the mean lifetimes of the two possible states of $T(t)$, then

$$R_T(\tau) = \frac{\tau_1^2}{(\tau_1+\tau_2)^2} + \frac{\tau_1\tau_2}{(\tau_1+\tau_2)^2}\exp\left(-\frac{\tau}{\tau_m}\right),$$

$$R_{\bar{T}}(\tau) = \frac{\tau_2^2}{(\tau_1+\tau_2)^2} + \frac{\tau_1\tau_2}{(\tau_1+\tau_2)^2}\exp\left(-\frac{\tau}{\tau_m}\right),$$

where $1/\tau_m = 1/\tau_1 + 1/\tau_2$. It follows that the power spectrum $S(\omega)$ of $V(t)$ is given by

$$S(\omega) = \frac{1}{(\tau_1+\tau_2)^2}\left[\tau_1^2 S_1(\omega) + \tau_2^2 S_2(\omega) + 2\tau_1\tau_2 \frac{\tau_m}{1+\omega^2\tau_m^2} * (S_1 + S_2)\right],$$

where the symbol * denotes the convolution product.

To see the implications of this result consider the following. If the motion on each attractor is periodic, then $S_1(\omega)$ and $S_2(\omega)$ are composed of delta functions at $n\omega_0$ where ω_0 is the driving frequency. If in addition $\tau_m \gg \omega_0^{-1}$, the resulting spectrum $S(\omega)$ will be composed of slightly broadened lines with Lorentzian line shape of width τ_m^{-1} and, providing the average values of $V_1(t)$ and $V_2(t)$ are different [in which case either $S_1(\omega)$ or $S_2(\omega)$ has a dc component], a low-frequency divergence in $S(\omega)$ going as ω^{-2} that saturates at $\omega \simeq 1/\tau_m$. This situation appears to be exactly the one depicted in Fig. 3 of Ref. 1 except that the ω^{-2} divergence at low frequencies has not yet saturated. {Note that the slope in the log-log plot of this figure in Ref. 1 is 20 dB/decade, which by the conventional definition of dB in terms of signal power [$\mathrm{dB} = 10\log S(\omega)$] corresponds to a $1/f^2$ law, not $1/f$ as concluded by the authors of Ref. 1.} If, on the other hand, $\tau_m \simeq \omega_0^{-1}$, the contributions $S_1(\omega)$ and $S_2(\omega)$ will be mixed, as is apparently the case in Fig. 2 of Ref. 1. Under these circumstances any range of frequencies for which $S(\omega)$ falls off as $1/\omega$ would be accidental and of no particular fundamental significance. Similar considerations apply if the motion on each attractor is chaotic.

In conclusion, although our model leaves out details possibly important in a fully quantitative theory such as subtle correlations between $T(t)$ and $V_1(t)$ or $V_2(t)$ (Ref. 3) or important transients during the passage between the two attractors,[5] we believe that it correctly accounts for the origins of the low-frequency part of the spectrum and that there is no fundamental connection to $1/f$ noise.

M. R. Beasley
Department of Applied Physics
Stanford University
Stanford, California 94305

D. D'Humieres
B. A. Huberman
Department of Applied Physics
Stanford University, Stanford, California 94305,
and Xerox Palo Alto Research Center
Palo Alto, California 94304

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