Nonlinear Stochastic Processes Driven by Colored Noise: Application to Dye-Laser Statistics

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The role of multiplicative colored noise on the photon statistics of the dye-laser output is investigated. This model explains consistently the recent experimental results by Short, Mandel, and Roy.

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The physics of macroscopic systems can often be described in terms of a few collective variables whose equations of motion are obtained by the elimination of microscopic variables and are, in general, highly nonlinear. If the system is in stable equilibrium, the statistical nature of the processes arising due to the microscopic dynamics can be neglected. However, in the case where the system approaches the limit of stability, the fluctuations due to microscopic variables play a very crucial role in governing the evolution of the system and hence must be incorporated into the equation of motion for macroscopic variables. The fundamental importance of fluctuations has been realized in a large number of systems, e.g., the laser, autocatalytic reactions, hydrodynamic and current instabilities, and equilibrium phase transitions for which the deterministic macroscopic description is found to be inadequate, especially near the threshold regime.

There is currently intense activity in analyzing the effect of fluctuations on the dynamics of nonlinear processes.[1] These fluctuations, appearing as additive and/or multiplicative[2] noise terms in the equations of motion, are usually modeled at δ-correlated Gaussian processes (or white noise). This assumption, although convenient for mathematical analysis, is somewhat unrealistic as the fluctuations arising due to the microscopic dynamics would have a finite (nonzero) correlation time. (Such fluctuations are commonly referred to as colored noise.) Only in certain regions of parameter space, where all other relevent times in the problem are much longer than the correlation time of the fluctuations, would the white-noise approximation be valid while in other regions discrepancies between white-noise theory and experiment would become noticeably large.

The problem of photon statistics of the dye-laser output falls into this category as the equation of motion for the electric field $\epsilon(t)$ is nonlinear and the colored noise is important. Recent experiments[3,4] have demonstrated some very interesting statistical properties of $\epsilon(t)$; e.g., the relative intensity fluctuations, defined as $\langle(\Delta I)^2\rangle/\langle I\rangle^2$, tend to increase indefinitely as the laser is weakly excited ($\langle I\rangle \to 0$). Such a behavior, inexplicable in terms of additive noise, is suspected to arise due to fluctuations in the pump parameter that appear multiplicatively in the equation of motion for $\epsilon(t)$. The theory proposed by Graham, Höhnerbach, and Schenzle,[5] where the noise was modeled as a Gaussian δ-correlated (white-noise) process, although successful in explaining some of the experimental results of Ref. 4, fails to explain results in another parameter regime as has been demonstrated by Short, Mandel, and Roy[4] in a recent publication.

It is the purpose of this Letter to explain the experimental results of Ref. 4 consistently by developing a theory that allows for the finite correlation time of the fluctuations (colored noise) in the pump parameters. The evolution of the dye-laser output is simulated numerically, as analytical solutions to this problem are difficult to obtain at this time.

We begin with the well-known equation of motion for the output of the single-mode dye laser operating in resonance,[6]

$$d\epsilon(t)/dt = [a_1(t) - A_1|\epsilon(t)|^2]\,\epsilon(t), \qquad (1)$$

where $\epsilon(t)$ (complex) is the field amplitude and A_1 provides for stabilization above threshold. The pump parameter $a_1(t)$ $[=a_{10}+\xi(t)]$ is considered to be fluctuating around its (real) mean value a_{10} with complex $\xi(t)$ $[=\xi_1(t)+i\,\xi_2(t)]$ denoting the fluctuations. Unlike in Ref. 5, where $\xi_1(t)$ and $\xi_2(t)$ were modeled as white noise, we assume these to be described by Ornstein-Uhlenbeck processes. $\xi_1(t)$ and $\xi_2(t)$ then obey the equations

$$d\xi_\kappa/dt = -\gamma\xi_\kappa(t) + \gamma F_\kappa(t), \quad \kappa = 1, 2, \qquad (2)$$

where $F_\kappa(t)$ are the Gaussian white-noise fluctuations satisfying

$$\langle F_i(t)\rangle = 0,$$
$$\langle F_i(t)F_j(t')\rangle = \delta_{ij} Q_{11}\delta(t-t'). \tag{3}$$

Equations (2) and (3) imply that

$$\langle \xi^*(t)\xi(t')\rangle = Q_{11}\gamma e^{-\gamma|t-t'|}, \tag{4}$$

thereby introducing a finite correlation time (γ^{-1}) for the pump parameter fluctuations. In the limit $\gamma\to\infty$, our model reduces to that of Graham, Höhnerbach, and Schenzle and that of Schenzle and Brand.[7]

Equations (1)–(3) are normalized[8] by defining $\tau = Q_{11}t$, $\epsilon(t) = (a_{10}/A_1)^{1/2}E(t)$, $\varphi_j(t) = \xi_j(t)/Q_{11}$, $\eta_j(t) = F_j(t)/Q_{11}$, $\alpha = a_{10}/Q_{11}$, $q = A_1/Q_{11}$, and $\Gamma = \gamma/Q_{11}$. The normalized equations are then written as

$$dE/d\tau = \alpha[1-|E|^2]E + \varphi E, \tag{5}$$

$$d\varphi/d\tau = -\Gamma\varphi + \Gamma\eta, \tag{6}$$

where real and imaginary parts of $\eta(\eta_1 + i\eta_2)$ satisfy

$$\langle \eta_i(\tau)\eta_j(\tau')\rangle = \delta_{ij}\delta(\tau-\tau'). \tag{7}$$

The average intensity of the dye-laser output is given by

$$\langle I\rangle = \langle|\epsilon(t)|^2\rangle = \alpha/q\langle|E(t)|^2\rangle = \alpha/q\langle \overline{I}(t)\rangle \tag{8a}$$

while the intensity correlation function is expressed as

$$\lambda(\tau) = \frac{\langle \Delta I(t_1)\Delta I(t_1+t)\rangle}{\langle I(t_1)\rangle^2} = \frac{\langle \Delta\overline{I}(\tau_1)\Delta\overline{I}(\tau_1+\tau)\rangle}{\langle \overline{I}(\tau_1)\rangle^2}. \tag{8b}$$

Thus $\lambda(\tau)$ depends only on α and Γ and q simply scales the absolute value of the intensity.

The solution of Eqs. (5) and (6) is simulated by extending the method of Sancho *et al.*[9] to the case of complex variables. The equations used in this simulation, to order Δ, are

$$E(\tau+\Delta) = E(\tau) + [1-|E|^2]E\alpha\Delta + E\varphi\Delta + \tfrac{1}{2}E\varphi^2\Delta^2, \tag{9}$$

$$\varphi_j(\tau+\Delta) = \varphi_j(\tau) - \Gamma\Delta\varphi_j(\tau) + \Gamma\Delta^{1/2}Z_\kappa, \quad \kappa = 1, 2, \tag{10}$$

where Z_1 and Z_2 are Gaussian random variables of zero mean and unit variance and are independent of each other. The last term in (9) is included so that the white-noise results are correctly reproduced up to Δ. Results presented in this paper have been obtained by averaging over 20 000 time steps of duration $\Delta = 0.005$ and over 350 different configurations. The maximum error bar in our numerical simulation is estimated to be about 10%.

A comparison of the results of our simulation for $\lambda(\tau)$ with the experimental results of Short, Mandel, and Roy[4] is made in Figs. 1 through 3 where the latter have been plotted on the normalized time scale ($\tau = Q_{11}t$) with $Q_{11} = 0.2$ (μsec)$^{-1}$. A constant background has been subtracted in fitting the data as was done in Refs. 4 and 5. Note that $\lambda(\tau)$ depends only on α and Γ. Unlike in the white-noise case ($\Gamma\to\infty$), where α is determined using $\lambda(0) = 1/\alpha$, complete knowledge of $\lambda(\tau)$ is required to determine α and Γ in the colored-noise case. Moreover, fluctuations of the pump parameter are characterized by Q_{11} and Γ and would be expected to remain constant at a fixed operating wavelength of the laser. As the operating wavelength was fixed in the experiment of Short, Mandel, and Roy,[4] Q_{11} and Γ would be expected to remain fixed for the experimental data plotted in Figs. (2)–(4) of Ref. 4 (hereafter referred to as cases I, II, and III, respectively). On the other hand, α, characterizing the proximity of the excitation to the threshold, would be different in the three cases. Therefore, we seek to determine a single value of Γ and three values of α (one for each case) to fit the experimental data.

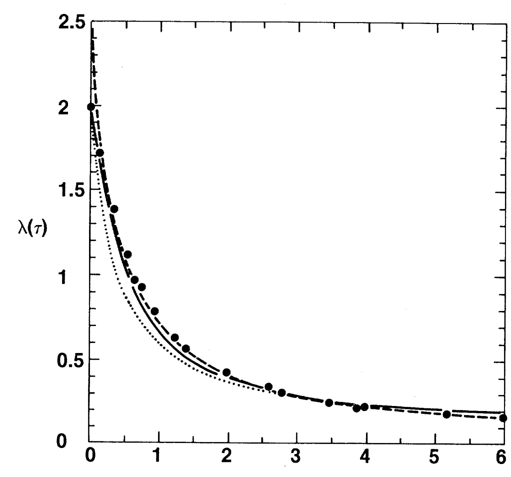

FIG. 1. $\lambda(\tau)$ vs τ for case I. Solid line, experimental data from Fig. 2 of Ref. 4; dotted line, white-noise theory with $\alpha = 0.52$; dashed line, white-noise theory with $\alpha = 0.4$; and dots, colored-noise theory with $\alpha = 0.4$ and $\Gamma = 5$.

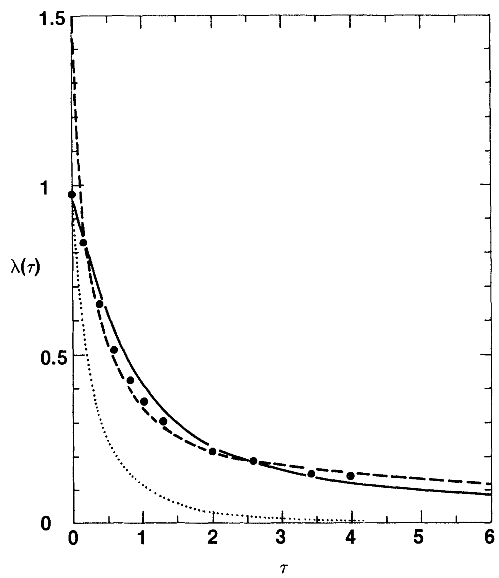

FIG. 2. $\lambda(\tau)$ vs τ for case II. Solid line, experimental data from Fig. 3 of Ref. 4; dotted line, white-noise theory with $\alpha = 1.05$; dashed line, white-noise theory with $\alpha = 0.82$; and dots, colored-noise theory with $\alpha = 0.82$ and $\Gamma = 5$.

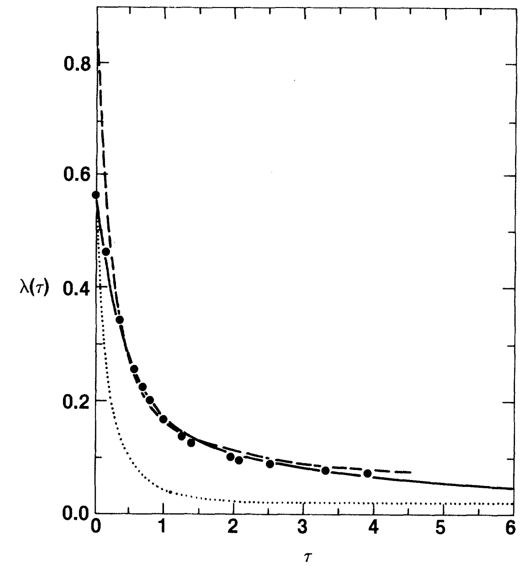

FIG. 3. $\lambda(\tau)$ vs τ for case III. Solid line, experimental data from Fig. 4 of Ref. 4; dotted line, white-noise theory with $\alpha = 1.84$; dashed line, white-noise theory with $\alpha = 1.3$; and dots, colored-noise theory with $\alpha = 1.3$ and $\Gamma = 5$.

Best fits are obtained for $\Gamma = 5$ for all three cases and $\alpha = 0.4$, 0.82, and 1.3 for cases I, II, and III, respectively. Within the accuracy of our simulations, the agreement between theory (dots) and experiment (solid lines) is excellent.

In Figs. 1, 2, and 3 we have also plotted, respectively, the white-noise-theory results (dashed curves) for $\alpha = 0.4$, 0.82, and 1.3. While these results agree with the results of our simulation for τ beyond 0.5 or so, there is considerable disagreement between the two results close to the origin. This is understandable with the realization that for $\tau \gtrsim 1$ the correlations in the pump fluctuations are vanishingly small (since $\Gamma\tau \gg 1$ for $\Gamma = 5$) and hence white-noise approximation should be valid in this region. For times such that $\Gamma\tau \lesssim 1$, finite correlation time of the fluctuations gives rise to significant deviations from the white-noise theory. It is seen that decreasing Γ reduces $\lambda(0)$ for a fixed α. This is due to the fact that the rate of random on-off switching of the laser is slower in the presence of colored noise as compared with the rate in the presence of white noise. This, in turn, reduces the fluctuations in the intensity output of the laser, thereby reducing $\lambda(0)$.

The implication of the above interpretation is the following: For $\Gamma > \alpha$ the tail part of the correlation function [$\lambda(\tau)$ for τ satisfying $\Gamma\tau \gg 1$] is identical to that obtained from the white-noise theory. However, $\lambda(\tau)$ for τ satisfying $\Gamma\tau \ll 1$ and $\alpha\ \tau \ll 1$ is sensitive to the correlation time of the fluctuations. Since $\lambda(0)$ for finite Γ is smaller than that for the white noise, a determination of α using the value of $\lambda(0)$ and the white-noise theory [$\alpha = 1/\lambda(0)$] will yield a higher value of α. Furthermore, $\lambda(\tau)$ calculated using the white-noise theory with this value of α shows considerable disagreement with the experimental results as seen in Figs. 2 and 3 (dotted lines and solid lines).

For case I, the white-noise theory of Graham, Höhnerbach, and Schenzle[5] also agrees with the experimental results. It is well known that close to threshold (small α), fluctuations of the light field slow down because of the critical slowing down process. In that case, the pump fluctuations (characterized by the correlation time Γ^{-1}) are much faster than the growth rate of the field (determined by α) and hence can be modeled as white noise. With increasing α the two time scales become comparable and hence the white-noise approximation breaks down. The success of the present model in explaining the observed features of $\lambda(\tau)$ even for large α justifies its appropriateness in describing the fluctuations. Other features of this model will be examined in a future publication.[10]

In conclusion, through numerical simulation of the dye-laser output in the presence of pump fluctuations characterized by a colored noise, we have been able to explain consistently the experimental results of Short, Mandel, and Roy.[4] Along with the usefulness of the numerical simulation, the results of the present paper also demonstrate the importance of the correlation time of the fluctuations on the dynamics of nonlinear stochastic processes. Effects such as the ones discussed here should be of importance in other nonlinear processes, for example, optical bistability, and are currently under investigation.

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[1] *Stochastic Systems in Physics, Chemistry and Biology*, edited by L. Arnold and R. Lefever (Springer, Berlin, 1981); *Dynamics of Synergetic Systems*, edited by H. Haken (Springer, Berlin, 1981).

[2] Multiplicative noise corresponds to those fluctuations that couple to the state of the system.

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[8] These normalization conditions follow naturally in transforming Eqs. (2)–(4) into Eqs. (5)–(7). The definitions of α and τ are consistent with those of Refs. 4 and 5. Furthermore, the definition of q used in this paper also follows from the analysis of Ref. 7, Sec. VI E, while it is clear that q as defined in Refs. 4 and 5 is inverted. However, the results for $\lambda(\tau)$ remain unaffected, as it is independent of q.

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