Large-N Quantum Chromodynamics at Finite Temperature

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It is pointed out that at $N\to\infty$, for finite temperature, the Schwinger-Dyson equations imply that below the deconfining phase transition the Wilson loops are independent of the temperature. This suggests a first-order deconfinement phase transition.

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It is commonly believed that non-Abelian gauge theories undergo a phase transition at finite physical temperature. Below the critical temperature the theory exhibits confinement of quarks and gluons; above the transition one is dealing with a gas of free gluons and quarks. This transition is supposed to be present also in the pure gauge theory. In such a theory there are no dynamical quarks but the transition can be studied by looking at the free energy of external static charges. In this Letter we discuss such a transition in a pure U(N) gauge theory in the limit of infinite N. Our main result is that *all* closed Wilson loops, both timelike and spacelike, are independent of the temperature in the confining phase. Finally we argue that the transition should be of first order by showing that the coefficient of the area law for large timelike closed loops jumps from a constant value to zero. Our argument makes use of the Schwinger-Dyson equations.

The Schwinger-Dyson equations for the expectation values of the Wilson loops have been derived and discussed in a number of papers.[1] Although solving these equations, even for $N\to\infty$, is a formidable task, they have been useful in obtaining general results about the theory.[2]

We will derive such equations for a U(N) gauge theory at finite temperature, and give a short discussion of some implications. The properties of a gauge theory at finite temperature can be derived by performing the functional integral over Euclidean field configurations that are periodic in the "time" variable.[3,4] We will study a lattice version of the theory with discrete time. The finite-temperature condition is

$$U_{n,\mu}=U_{n+n_T\hat{t}_0,\mu}\,, \tag{1}$$

where $\hat{t}_0$ is a unit vector in the time direction and $n=(n_0,\vec{n})$ denotes a lattice site. $U_{n,\mu}$ is the $N\times N$ unitary matrix corresponding to the link joining the site n and $n+\mu$ where the temperature is given by

$$\beta=(KT)^{-1}=n_T a\,, \tag{2}$$

where a is the lattice spacing. The statistical mechanics of the system follows from the partition function

$$Z=\prod_{\vec{n}=-\infty}^{\infty}\prod_{n_0=0}^{n_T-1}\prod_{\mu=1}^{d}\left(\int dU_{n,\mu}\right)e^{-S}, \tag{3}$$

$$S=-\sum_{\vec{n}=-\infty}^{\infty}\sum_{n_0=0}^{n_T-1}\sum_{\mu\neq\nu=1}^{d}\frac{N}{\lambda}\operatorname{tr}(U_{n,\mu}U_{n+\mu,\nu}U_{n+\nu,\mu}{}^{\dagger}U_{n,\nu}{}^{\dagger}). \tag{4}$$

Here d is the dimension of space-time. We assume that the lattice is infinite in the space directions and has a size n_T in the Euclidean time. The expectation value of a Wilson loop

$$\varphi(C)=\frac{1}{N}\operatorname{tr}\prod_{l\in C}U(l) \tag{5}$$

is

$$W(C)=\int(dU)e^{-S}\,\varphi(C)/Z. \tag{6}$$

Here $(dU)\equiv\prod_n dU_{n,\mu}$, C is a closed contour in space-time, and l in (5) is a link on C. The Schwinger-Dyson equations are derived by considering the quantity

$$\langle\operatorname{tr}(U_{n,\mu}\cdots U_{m-\nu,\nu}T^jU_{m,\alpha}U_{m+\alpha,\delta}\cdots U_{n-\epsilon,\epsilon})\rangle, \tag{7}$$

where T^j is a generator of $U(N)$. Making the change of variables

$$U_{m,\nu} \to \exp(i\epsilon T^j) U_{m,\nu} \tag{8}$$

on the link m, ν does not change the expectation value of the operator in Eq. (7). Therefore the terms of order ϵ must be zero. By use of the identities

$$U_{n,\nu} = U_{n+\nu,-\nu}{}^{\dagger}, \tag{9}$$

$$\sum_j T_{kl}{}^j T_{mn}{}^j = \delta_{kn}\delta_{ml}, \tag{10}$$

and the periodicity condition of Eq. (1), the resulting Schwinger-Dyson equation can be written as

$$\lambda^{-1} d(n(\tau), \nu(\tau)) W(C) = \sum_{\tau'=1}^{L(C)} \sum_{k=-\infty}^{\infty} \delta(n(\tau), \nu(\tau)|n(\tau') + kn_\tau \hat{i}_0, \nu(\tau')) W(C_{\tau\tau'}) W(C_{\tau'\tau}), \tag{11}$$

where $1 \leq \tau \leq L(C)$ and $1 \leq \tau' \leq L(C)$ are (discrete) parameters on the loop, and $L(C)$ is the number of links in C. $n(\tau)$ and $\nu(\tau)$ are, respectively, the origin and the direction of the τ link in C. $\hat{i}_0$ is a unit vector in the time direction. $d(n, \nu)$ is the Makeenko-Migdal derivative on the Wilson loop, corresponding to the replacement

$$U_{n,\nu} \to \sum_{\substack{\mu\neq\nu \\ \mu\neq-\nu}} (U_{n,\mu}U_{n+\mu,\nu}U_{n+\mu+\nu,-\mu} - U_{n,\nu}U_{n+\nu,\mu}U_{n+\nu+\mu,-\nu}U_{n+\mu,-\mu}U_{n,\nu}). \tag{12}$$

The expression

$$\delta(n, \nu|m, \mu) = \delta_{nm}\delta_{\nu\mu} - \delta_{nm+\mu}\delta_{\nu-\mu} \tag{13}$$

is the "link" delta, which is zero unless the links n, ν and m, μ join the same points. When they do join the same points, it is $+1$ if they go in the same direction and -1 if they go in opposite directions. $C_{\tau\tau'}$ and $C_{\tau'\tau}$ are the two loops obtained by splitting C at the points τ and τ'. To derive Eq. (11) we have used the Migdal factorization condition

$$\langle N^{-1}\operatorname{tr}(\Pi U) N^{-1}\operatorname{tr}(\Pi' U)\rangle = \langle N^{-1}\operatorname{tr}(\Pi U)\rangle \langle N^{-1}\operatorname{tr}(\Pi' U)\rangle. \tag{14}$$

The usual Schwinger-Dyson equation corresponds to the term with $k=0$ in Eq. (11). The additional terms appearing on the right-hand side of the equation correspond to the possibility of breaking a closed loop into two loops that are closed only as a result of the periodicity in time (see Fig. 1). Such "open" Wilson loops are equal to $e^{-\beta f}$, where f is the free energy of free static quarks. Therefore they vanish in the confining phase. Hence, in the confining phase, the Schwinger-Dyson equations are the same at finite temperature as they are at zero temperature. The conclusion is that, in the confining phase and in leading order of $1/N$, the Wilson loops are independent of the temperature. Note that this is a strictly nonperturbative result. To any finite order in weak-coupling perturbation theory the closed loops are temperature dependent since the open loops are nonzero to any finite order. Since the Wilson loops are independent of the temperature the coefficient of the area law is also constant in the confining phase. On the other hand, it is easy to convince oneself that the coefficient of the area for large timelike Wilson loops is zero above the deconfinement phase transition. A brief "proof" goes as follows. Consider a rectangular timelike loop extending Mn_τ lattice sites in the time direction and x sites in the space direction. This way the top spacelike edge of our rectangular loop lies—because of periodicity in time—on top of the lower spacelike edge and traverses it in the opposite direction. If we perform the Makeenko-Migdal derivative at a spacelike link, the loop will break—by virtue of factorization to leading order—into two open loops. But those open loops are nonzero and, because of the "backtracking" explained above, independent of x. Therefore, in order to have a consistent equation the coefficient of the area for large timelike loops must be zero. That open loops of all lengths are nonzero can be shown by use of a similar argument. We feel that since the coefficient of the area for large timelike loops is zero above the deconfining phase transition and constant (and finite) in the confining phase, the

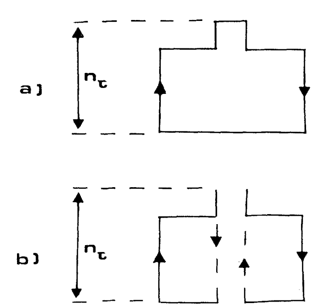

FIG. 1. The loop in (a) can split into the "loops" in (b) as a result of periodicity in the time variable.

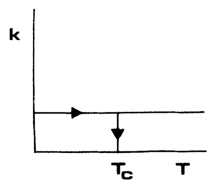

FIG. 2. The coefficient of the area of large timelike loops as a function of temperature.

coefficient must have a discontinuity at the transition (see Fig. 2). This discontinuity is only possible if the deconfining transition is a first-order phase transition.[5,6]

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