Multimode Theory of Free-Electron Laser Oscillators

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The nonlinear self-consistent theory of short-pulse free-electron laser oscillators can now be extended to include transverse diffraction within the optical resonator mode. The theory provides efficient solutions to the three-dimensional parabolic-wave equation coupled to the Lorentz force equation. The method is general enough to include arbitrary magnet designs, optical mirror arrangements, and driving currents. New mode structures are predicted which should be observed in future experiments at Stanford University.

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Optical wave fronts in a free-electron laser (FEL) are driven by relativistic electrons traveling through a static, transverse undulating magnetic field.[1,2] In the oscillator these wave fronts are stored in an open spherical-mirror resonator.[3] Typically the shape of the electron beam from an accelerator or storage ring does not match the longitudinal or transverse mode structure of the optical resonator (see Fig. 1) so that a theoretical analysis must be capable of describing the nonlinear multimode behavior that develops over many passes. The first measurements of these higher-order modes will be made in the next year at the Stanford University High Energy Physics Laboratory; their initial experiments are in agreement with our theory.[4]

Since the original FEL experiments, numerous studies have been made that have been restricted to a single-mode plane-wave analysis.[5,6] The longitudinal mode structure in an FEL with a short optical pulse has been examined experimentally[7] and theoretically.[8-10] The transverse modes of the FEL have also been considered,[11-16] but calculations have been restricted to the single-pass amplifier without a self-consistent electron current. The full nonlinear, multimode problem of the FEL oscillator with a self-consistent current remained to be solved.

The FEL system is properly described by the coupled Maxwell and Lorentz-force equations. Here we present a powerful self-consistent method of solution to the nonlinear FEL oscillator problem describing the complex mode evolution in each dimension. The undulator magnet through which electrons pass is $\vec{B}=[B\cos k_0z, B\sin k_0z, 0]$ where B is the magnetic field amplitude which extends over a length L, and $\lambda_0=2\pi/k_0$ is the magnet wavelength. With perfect injection into helical orbits the electron's velocity is $c\vec{\beta}=-[\beta_t \times\cos k_0z, \beta_t\sin k_0z, \beta_0]c$ where $\beta_t=eB\lambda_0/2\pi\gamma mc^2$, $e=|e|$ is the magnitude of the electron's charge, m is the electron's mass, c is the speed of light, and γmc^2 is the electron's energy.

The optical vector potential is general in form except that the polarization is chosen to match the spontaneous emission from electrons in the undulator so that $\vec{A}(\vec{x}, t)=k^{-1}|E(\vec{x}, t)|[\sin\psi, \cos\psi, 0]$ where $\psi=kz-\omega t+\varphi(\vec{x}, t)$, $\omega=kc$, and $\lambda=2\pi/k$ is the optical wavelength. The complex electric field envelope $E(\vec{x}, t)=|E(\vec{x}, t)|e^{i\varphi(\vec{x}, t)}$ is taken to be slowly varying in z and t so that when $\vec{A}(\vec{x}, t)$ is inserted into Maxwell's equations the result is the well-known parabolic wave equation. In a dimensionless coordinate system where $\tau=tc/L$, $\bar{x}=x(k/2L)^{1/2}$, $\bar{y}=y(k/2L)^{1/2}$, and $\bar{z}=(z-ct)/\delta$ with δ being the electron's pulse length, the parabolic wave equation becomes

$$(-\tfrac{1}{4}i\,\bar{\nabla}_t^{\,2}+\partial/\partial\tau)a(\bar{x},\bar{y},\bar{z},\tau) = \langle je^{-i\zeta}\rangle_{(\bar{x},\bar{y},\bar{z}+s\tau,\tau)}, \qquad (1)$$

where $\bar{\nabla}_t^{\,2}=\partial_{\bar{x}}^{\,2}+\partial_{\bar{y}}^{\,2}$, and the dimensionless electric field is $a=4\pi Ne\beta_tLE/\gamma_0mc^2$. The current density is $j=8\pi^2Ne^2\beta_t^{\,2}L^2\rho/\gamma_0mc^2$, γ_0mc^2 is the electron's initial energy, the slippage is $s=(1-\beta_0)L/\delta$, and angular brackets denote an average over the electron phases $\zeta=(k+k_0)z-\omega t$. Equation (1) governs the dynamics of the optical wave over many optical wavelengths in the longitudinal (z) dimension[17] and the $\bar{\nabla}_t^{\,2}$ operator properly describes the diffraction of the optical wave in the (x, y) directions. The result (1) without diffraction (without the $\bar{\nabla}_t^{\,2}$ term) has been derived earlier[17] and used to solve the FEL problem of short-pulse propagation.[8]

The incremental solution to (1) is given by

$$a(\bar{x},\bar{y},\bar{z},\tau+\Delta\tau)=\exp(\tfrac{1}{4}i\Delta\tau\bar{\nabla}_t^2)a(\bar{x},\bar{y},\bar{z},\tau)-\Delta\tau\langle je^{-i\zeta}\rangle_{(\bar{x},\bar{y},\bar{z}+s\tau,\tau)}+O(j\Delta\tau^2), \quad (2)$$

and this solution is exact when there is no current $j=0$. To evaluate the diffraction operator $\exp(i\Delta\tau\bar{\nabla}_t^2/4)$ we work in Fourier space where the operator is diagonal and may be efficiently implemented numerically.[18] Wave fronts $a(\bar{x},\bar{y})$ can now be correctly propagated forward in time and the current will amplify the wave where $j(\bar{x},\bar{y})$ is nonzero. When the transverse size of the electron pulse and the wave fronts are much larger than $(2L/k)^{1/2}$, then the operator $\exp(i\Delta\tau\bar{\nabla}_t^2/4)\approx 1$ and diffraction can be neglected.

The self-consistent evolution of the electron current in (1) is governed by the Lorentz-force equation $\dot{\gamma}=e\vec{\beta}\cdot(\partial\vec{A}/\partial t)/mc^2$ for each electron in the beam. With use of $\vec{\beta}$ and $\vec{A}(\vec{x},t)$ given above, the electron equation of motion has been previously shown to take the form of the well-known pendulum equation in ζ,

$$\dot{\nu}(\bar{x},\bar{y},\bar{z}+s\tau,\tau)=\ddot{\zeta}(\bar{x},\bar{y},\bar{z}+s\tau,\tau)=\tfrac{1}{2}[a(\bar{x},\bar{y},\bar{z},\tau)e^{i\zeta(\bar{x},\bar{y},\bar{z}+s\tau,\tau)}+\text{c.c}], \quad (3)$$

where the overdot means derivative with respect to τ and $\nu=L[(k_0+k)\beta_z-k]=\dot{\zeta}$ is the dimensionless velocity for each electron. The initial electron coordinates (ζ_0,ν_0) are arbitrary, but typically they are chosen to be uniformly spread in ζ_0 and monoenergetic in ν_0 to characterize a realistic FEL.

The coupled equations (2) and (3) form our theory. They have been used extensively in single-mode calculations[14,17] [where $\zeta(\tau)$ does not depend on $(\bar{x},\bar{y},\bar{z})$], and in longitudinal multimode calculations[8] where transverse effects, $(\bar{x},\bar{y})$ dependence, were negligible. As in Ref. 8 short-pulse effects are included through the electron pulse length δ in s; the extension here is to include all three spatial modes $(\bar{x},\bar{y},\bar{z})$. When (2) and (3) are integrated together, each electron responds to the superimposed optical field at each site $(\bar{x},\bar{y},\bar{z})$ at each time τ, and the method is explicitly self-consistent and nonlinear. The examples we have chosen include diffractive effects in $(\bar{x},\bar{y})$ but not the better understood slippage problem,[8-10] so that $s=0$ in what follows.

The method used to determine the stable resonator modes in the FEL oscillator is similar to the method used by Fox and Li[19] to determine the eigenvalues of optical resonators without sources. Our procedure is to propagate the optical wave from mirror to mirror along the undulator where it is amplified by the electron beam as shown in Fig. 1. Both mirrors $M1$ and $M2$ have radius r_m in dimensionless transverse units [the mirror radius times $(k/2L)^{1/2}$]. The first mirror $M1$ is taken to be perfectly reflecting while the fractional power lost through $M2$ is $e^{-1/Q}$ per pass. At each mirror surface, the wave front experiences a phase shift given by $\delta\varphi(\bar{x},\bar{y})=-\bar{r}^2/r_c$ where $\bar{r}^2$

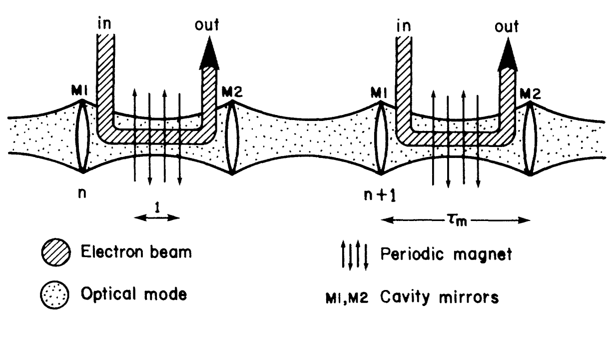

FIG. 1. The resonator cavity unfolded in time to follow the evolution of the optical wave front. In this view the mirrors $M1$ and $M2$ act as lenses to focus the wave front back to the mode axis. The wave front is amplified by a new electron beam on each pass through the laser's undulating magnetic field.

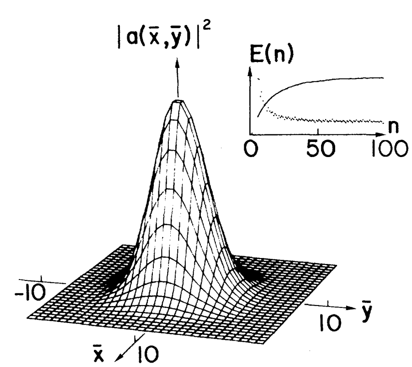

FIG. 2. The steady-state wave front $|a(\bar{x},\bar{y})|^2$ at the output mirror $M2$ after $n=100$ passes resembles the TEM_{00} cavity mode. The wave-front energy $E(n)$ (inset curve) has increased to a final value of $E(100) = 2600$.

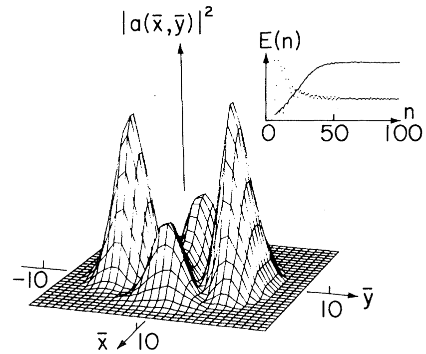

FIG. 3. The steady-state wave front $|a(\bar{x},\bar{y})|^2$ at the output mirror $M2$ after $n = 100$ passes. The off-axis electron beam with $\Delta\bar{y} = 1.0$ has distorted the steady-state wave front into four distinct peaks. The peak optical field $|a|_{max} = 15$ and the wave-front energy is increased to a final value of $E(100) = 1100$.

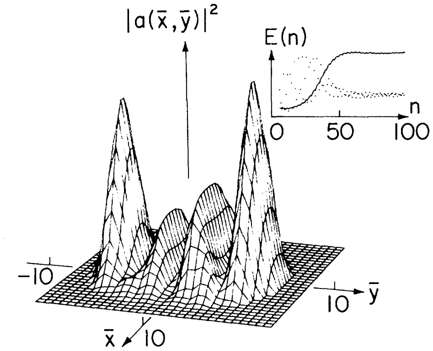

FIG. 4. With the electron beam further off axis we find another steady-state wave front with weaker fields $|a|_{max} = 10$. This mode has a structure reminiscent of the TEM_{30} mode and the wave-front energy is now $E(100) = 500$.

$=\bar{x}^2+\bar{y}^2$, and r_c is the radius of curvature of the mirror divided by L. The mirror separation divided by the undulator length L is τ_m. The wave fronts propagate freely outside the undulator and on the return trip from $M2$ to $M1$.

When entering the undulator the wave front is amplified by the electron beam. The electrons at each point $(\bar{x},\bar{y})$ are uniformly spread in the initial phases ζ_0 and all start at $\nu_0 = 2\pi$ for maximum gain in strong fields. Along the undulator we must integrate the coupled equations (2) and (3) from $\tau = 0$ to 1, the end of the magnet. The shape of the electron-beam density is arbitrary in the theory; our example used the parabolic form $j(\bar{r}) = j_0[1-\bar{r}^2/(2\bar{\sigma}^2)]$ for $\bar{r}^2 < 2\bar{\sigma}^2$ and $j = 0$ elsewhere outside the electron beam. The peak density is j_0 and the width is $\bar{\sigma}$.

We continue bouncing the light between the mirrors until steady state is attained. Amplification and mode distortion continue until the optical fields become strong enough to reduce the gain per pass to balance the resonator losses.[8]

The final mode structure then depends on the nonlinear properties of the combined electron beam and resonator described by the parameters j_0, $\bar{\sigma}$, Q, r_m, and r_c. The results displayed in Fig. 2 use a resonator described by $r_c = 1.4$, $r_m = 3.0$, $\tau_m = 2.4$, and $Q = 66$. The peak current density $j_0 = 60$ is spread over a width $\bar{\sigma} = 0.25$ centered on the resonator axis. Observe that the wave-front energy (inset curve) after the nth pass, $E(n) = \int d\bar{x}\, d\bar{y}\, |a(\bar{x},\bar{y})|^2$, grows until the strong optical fields reduce the gain to match the resonator losses. The gain $[E(n)/E(n-1) - 1]$ is shown as a dotted curve. The stable mode is reached with a peak field $|a(\bar{x},\bar{y})|_{max} = 24$ and differs slightly from the free case ($j = 0$). In this case the resonator operates with a mode structure that is qualitatively similar to the lowest-order free-cavity mode. These parameters have been chosen to simulate the Stanford FEL and recent experiments[4] are in agreement with the mode structure shown in Fig. 2.

More complicated mode structure is possible as shown in Figs. 3 and 4. In these cases, the electron beam travels parallel to, but off of, the mode axis by amounts $\Delta\bar{y} = 1.0 = 4\bar{\sigma}$ and $\Delta\bar{y} = 1.2 = 4.8\bar{\sigma}$, respectively. In each case the mode peak on the positive $\bar{y}$ axis overlaps the electron beam when in the undulator and spreads to the larger pattern shown at the $M2$ mirror surface. The result of the off-axis electron beam is lower power in the FEL oscillator. We expect this behavior to be seen in experiments that will soon be carried out at Stanford University.[4]

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