Comment on "Entropy Evaporated by a Black Hole"

Zurek[1] has estimated that the massless radiation evaporated by an uncharged, nonrotating black hole in vacuum will have $R \equiv dS_{\rm rad}/(-dS_{\rm BH}) \approx \frac{4}{3}$ times the entropy decrease of the hole. Here I wish to note that numerical calculations[2] have given somewhat larger values for R. One can even make R arbitrarily large by surrounding the black hole with a shell of arbitrarily low transmission Γ, or one can make R arbitrarily close to unity by allowing the hole to emit only very massive particles.

In Planck units, an uncharged, nonrotating black hole of mass M has a Hawking temperature[3] $T_{\rm BH} = (8\pi M)^{-1}$ and entropy $S_{\rm BH} = 4\pi M^2$ and emits noninteracting radiation in each mode with a density matrix[4,5]

$$\rho_{nn'} = \delta_{nn'}\Gamma^n(e^x \mp 1)^{\pm 1}(e^x \mp 1 \pm \Gamma)^{-n\mp 1}.$$

The upper sign is for bosons, which can have any nonnegative integral number n of quanta, and the lower sign is for fermions, which can only have $n=0$ or 1. Γ is the transmission coefficient of the hole to a particle of species s and rest mass μ in a mode of energy $\omega \equiv T_{\rm BH}x$, helicity p, and angular momentum (l, m). In the adiabatic semiclassical approximation, the black hole is assumed to have a nearly Schwarzschild classical metric with $M \gg 1$ which slowly decreases at the rate given by the expectation value of the radiation power, and all gravitationally mediated quantum correlations between the different radiation modes are ignored. Then calculation of the expected energy $\mathrm{tr}(-\rho n\omega)$ and the entropy $\mathrm{tr}(-\rho\ln\rho)$ in each mode and integration over all modes gives

$$-\frac{dS_{\rm BH}}{dt} = -\frac{1}{T_{\rm BH}}\frac{dM}{dt} = \frac{1}{2\pi}\sum_{s,p,l,m}\int_\mu^\infty d\omega\frac{\Gamma x}{e^x \mp 1},$$

$$\frac{dS_{\rm rad}}{dt} = \frac{1}{2\pi}\sum_{s,p,l,m}\int_\mu^\infty d\omega\left[\frac{\Gamma}{e^x \mp 1}\ln\left(\frac{e^x \mp 1}{\Gamma} \pm 1\right) \pm \ln\left(1 \pm \frac{\Gamma}{e^x \mp 1}\right)\right].$$

Insertion of numerical calculations of the Γ's into these integrals for massless neutrinos, photons, and gravitons gave[2]

$$-dS_{\rm BH}/dt \approx 10^{-3}M^{-1}(2.0566n_{1/2} + 0.8454n_1 + 0.0964n_2),$$

$$dS_{\rm rad}/dt \approx 10^{-3}M^{-1}(3.3710n_{1/2} + 1.2684n_1 + 0.1300n_2),$$

where n_s is the number of two-polarization species of spin s. For the hypothetical case of neutrino emission, $R_{1/2} \approx 1.6391$; for pure photon emission, $R_1 \approx 1.5003$; for pure graviton emission, $R_2 \approx 1.3481$; and for three massless neutrino species plus photons and gravitons, $R \approx 1.6187$. These numbers are generally significantly higher than Zurek's estimate because his approximation effectively assumes $\Gamma = 0$ or $\Gamma = 1$ for each mode. Indeed, if one surrounds the black hole with a shell at $M \ll r \ll M^{5/3}$ with a very small constant transmission coefficient Γ for all modes that intersect the shell, one gets $R \sim 30\pi^{-4}\ln\Gamma^{-1}$, which can be made arbitrarily large by making Γ arbitrarily small. Alternately, if all particles were massive with $M\mu \gg 1$, then $\Gamma \approx \theta(4M\omega - l)$ and $2s+1$ helicities give

$$-dS_{\rm BH/dt} \approx 8\pi^{-1}(2s+1)M^2\mu^3\exp(-8\pi M\mu),$$

$$R = dS_{\rm rad}/(-dS_{\rm BH}) \approx 1 + (8\pi M\mu)^{-1},$$

which is very nearly reversible.

This Comment was stimulated by correspondence and discussion with Wojciech Zurek, which unfortunately occurred too late for him to incorporate these results into his paper. The numerical refinements and further examples given here do not invalidate his main qualitative conclusions concerning Bekenstein's conjecture but show how they may be illustrated more precisely.

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