GIANT FLUCTUATIONS IN A DEGENERATE FERMI GAS*

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In two recent notes,[1,2] Overhauser has suggested that the ground state of an interacting gas of fermions has giant density or spin-density fluctuations. On the basis of his considerations, he has cast serious doubt on current theories of nuclear and metallic structure. We wish to present reasons for our view that even if such fluctuations occur in one dimension,[3] there is at present no reason to believe that they exist in two or three dimensions, provided the interaction is sufficiently weak.

Overhauser has pointed out the remarkable fact that both the Hartree and Hartree-Fock equations of a one-dimensional gas have solutions which give lower energies than the customary plane wave solutions. That for strong interactions such lower solutions exist had been well known: Wigner's electron crystal lattice[4] in the low-density (or strong-interaction) limit is an example. What makes Overhauser's conclusions so disturbing is that they apply also to systems with quite weak interactions, such as nuclear matter and metallic electron gases. (In these systems, although the interactions contribute substantially to the total energy, their effect on the dynamics of the particles is generally regarded as fairly small.) In the present paper, we put forward arguments for the view that for sufficiently weak interactions the Hartree (and Hartree-Fock) ground states in two or more dimensions are the familiar plane wave states.

For simplicity, we restrict our considerations to the case of Hartree's equations. Let the interaction between particles be given by

$$H' = \gamma \sum_{i>j} v(\vec{r}_i - \vec{r}_j), \tag{1}$$

where the $\vec{r}_i$ denote the positions of the one- or more-dimensional fermions. If we denote the density of the gas by $n(\vec{r})$, then the Hartree potential is

$$V(\vec{r}) = \gamma \int v(\vec{r} - \vec{r}\,')n(\vec{r}\,')d\vec{r}\,'. \tag{2}$$

Clearly, only if the density is nonuniform can we have a nonconstant Hartree potential and, hence, a nontrivial solution of Hartree's equations. For $\gamma = 0$, the ground state is the familiar one with $n(\vec{r}) = n_0$. For a small γ, it follows from continuity that in the Hartree ground state we have[5]

$$n(\vec{r}) = n_0 + \lambda n'(\vec{r}) + \cdots, \tag{3}$$

where we may normalize $n'(\vec{r})$ by the condition

$$\frac{1}{\Omega}\int [n'(\vec{r})]^2 d\vec{r} = n_0^2 \tag{4}$$

(Ω = volume of system), and where

$$\lim_{\gamma \to 0} \lambda = 0. \tag{5}$$

Hence, apart from an irrelevant constant,

$$V(\vec{r}) = \gamma\lambda U(\vec{r}), \tag{6}$$

where

$$U(\vec{r}) = \int v(\vec{r} - \vec{r}\,')n'(\vec{r}\,')d\vec{r}\,'. \tag{7}$$

Now we shall check the self-consistency. We solve the Schrödinger equations,

$$\left[-\sum_i \frac{\partial^2}{\partial x_i^2} + \gamma\lambda U(\vec{r})\right]\psi = E\psi, \tag{8}$$

and from the lowest solutions construct the new density, $\bar{n}(\vec{r})$. If for small values of $(\gamma\lambda)$, $\bar{n}(\vec{r})$ is linear in this parameter, we have

$$\bar{n}(\vec{r}) = n_0 + \gamma\lambda\bar{n}'(\vec{r}) + \cdots. \tag{9}$$

Comparison with Eq. (3) shows that this is a contradiction unless $\lambda = 0$, which is the trivial solution.

We must then examine the crucial question of the above-mentioned linearity. We shall make the assumption—to which we come back at the end of this paper—that the dependence of $\bar{n}$ on $(\gamma\lambda)$ is linear provided that first order perturbation theory on the unperturbed ground state gives a finite result.

Let us Fourier-analyze $U(\vec{r})$, which leads to the problem

$$\left[-\sum_i \frac{\partial^2}{\partial x_i^2} + \epsilon\cos(\vec{q}\cdot\vec{r})\right]\psi = E\psi. \tag{10}$$

To first order in ϵ the normalized solutions are

$$\varphi_k = \Omega^{-1/2}\left[e^{i\vec{k}\cdot\vec{r}} + \frac{\epsilon}{2}\left(\frac{e^{i(\vec{k}+\vec{q})\cdot\vec{r}}}{\vec{k}^2 - (\vec{k}+\vec{q})^2} + \frac{e^{-i(\vec{k}-\vec{q})\cdot\vec{r}}}{\vec{k}^2 - (\vec{k}-\vec{q})^2}\right)\right], \tag{11}$$

and the corresponding density change is

$$\delta\bar{n}(\vec{r}) = \frac{\epsilon}{(2\pi)^{\nu}}\left\{\int d\vec{k}\left[\frac{1}{\vec{k}^2 - (\vec{k}+\vec{q})^2} + \frac{1}{\vec{k}^2 - (\vec{k}-\vec{q})^2}\right]\right\}\cos(\vec{q}\cdot\vec{r}). \quad (12)$$

Here ν is the dimensionality of the particles, the integral over k extends over the unperturbed Fermi "sphere" and, if singularities occur in the integrand, principal values are to be taken.

In one dimension, the result is

$$\delta\bar{n}(x) = \frac{\epsilon}{2\pi}\left\{-\frac{1}{q}\ln\left|\frac{q^2+2k_F q}{q^2-2k_F q}\right|\right\}\cos(qx), \quad (13)$$

where k_F is the Fermi momentum. The coefficient of ϵ is finite except at $q = 2k_F$. We conclude that nontrivial self-consistent solutions can exist only for $q \to 2k_F$. (These are in fact just Overhauser's density fluctuations.)

On the other hand, in two dimensions Eq. (12) gives

$$\delta\bar{n}(\vec{r}) = (\epsilon/4\pi^2)F(q)\cos(\vec{q}\cdot\vec{r}), \quad (14)$$

where

$$F(q) = -\pi, \qquad q \leq 2k_F$$

$$= -\pi\{1 - [1 - (2k_F/q)^2]^{1/2}\}, \quad q \geq 2k_F. \quad (15)$$

Here the coefficient of ϵ remains finite even for $q = 2k_F$. The same is true in three dimensions.

We now come back to the basic assumption stated in the paragraph following Eq. (9). This has not been generally proved and remains the weak point in our considerations. However, it has been verified in the following exactly soluble cases:

1. One dimension, $U = \cos qx$. Here the exact $\bar{n}$ is, to leading order, linear in $(\gamma\lambda)$ except for $q = 2k_F$, where the exact $\delta\bar{n} \sim (\gamma\lambda)\ln(\gamma\lambda)$.

2. Two dimensions, $U = c_1\cos q_1 x + c_2\cos q_2 y$. Here the exact $\delta\bar{n}$ (calculated with a distorted Fermi surface) is linear in $(\gamma\lambda)$ for all q_1, q_2.

If q_1 or $q_2 = 2k_F$, the Fermi surface has a flat portion of length $\sim(\gamma\lambda)^{1/2}$ which contributes a nonanalytic term of the form $(\gamma\lambda)^{3/2}\ln(\gamma\lambda)$; this is, however, negligible compared to the first order term.

We have also studied approximately the polygonal Fermi surfaces suggested by Overhauser. As $(\gamma\lambda) \to 0$, the number of sides must rapidly approach ∞ if the energy is to be lower than in the trivial solution. However, because the dimension of the individual sides then approaches zero with $(\gamma\lambda)$, self-consistency again appears to be impossible.

Thus we conclude that for weak interactions the Hartree equations have a nontrivial ground state only in one dimension. Although we have not yet demonstrated it in detail, there is little doubt that the same is true of the Hartree-Fock equation.

We suggest that the quantitative successes of current theories of nuclear and metallic structure imply that in general the interactions are sufficiently weak so that the usual Hartree-Fock functions represent the correct lowest order approximation.

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[1]A. W. Overhauser, Phys. Rev. Letters 4, 415 (1960).

[2]A. W. Overhauser, Phys. Rev. Letters 4, 466 (1960).

[3]Even this is still in question, since in Overhauser's work on one-dimensional fermions correlation effects are not included.

[4]E. P. Wigner, Trans. Faraday Soc. 34, 678 (1938).

[5]In our situation of an extensive system $n'(\vec{r})$ may have to be considered as a function of λ; e. g., $n'(\vec{r}) = a\cos[\vec{q}(\lambda)\cdot\vec{r}]$. This dependence on λ is of no essential consequence and will not be explicitly indicated. A fuller discussion of this point will be given in a forthcoming paper.