EQUIVALENCE OF THE CRITICAL CONCENTRATIONS IN THE ISING AND HEISENBERG MODELS OF FERROMAGNETISM

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If a magnetic crystal is diluted but otherwise unchanged by the addition of nonmagnetic atoms randomly arranged in the lattice, the transition temperature falls. There is a certain concentration of magnetic atoms below which the system will not exhibit a transition to magnetic order. At this critical concentration the transition temperature is at 0°K. It is the purpose of this Letter to show that in ferromagnetic crystals the critical concentration is a topological property of the lattice and is the same in the Heisenberg and Ising models and is independent of the value of the spin S. Nearest neighbor interactions only are assumed so that the Hamiltonians are $-2J\sum_{\langle i,j\rangle}\vec{S}_i\cdot\vec{S}_j$ and $-2J\sum_{\langle i,j\rangle}S_i^zS_j^z$ for the Heisenberg and Ising models, respectively.

The result may be derived from a consideration of the behavior of the susceptibility χ when expanded in a power series in the concentration p of magnetic atoms. In the random dilute system let there be N_1 isolated magnetic atoms, N_2 isolated nearest neighbor pairs, ... N_n isolated connected clusters of n magnetic atoms. For $n>2$ there will be various types of cluster which are specified by α, say. N_n is therefore subdivided into the numbers $N_{n,\alpha}$ of each type. The susceptibility can be written as

$$\chi kT/4\beta^2=\sum_{n,\alpha}N_{n,\alpha}\langle[S^z(n,\alpha)]^2\rangle_T, \tag{1}$$

where the last bracket is the thermal average of the square of the z component of the total spin of the cluster. At any T this can be evaluated only if the energy levels of the cluster are evaluated, but at $T=0$°K it can be simply written down for any cluster.

In the Ising problem the lowest energy state has all the spins oriented in the same direction and

$$\langle[S^z(n,\alpha)]^2\rangle_0=n^2S^2. \tag{2}$$

In the Heisenberg problem the lowest state is the symmetric one of maximum total spin nS and

$$\langle[S^z(n,\alpha)]^2\rangle_0=\tfrac{1}{3}nS(nS+1). \tag{3}$$

The $N_{n,\alpha}$ can be expanded in powers of p, each one having p^n as its smallest power. Equation (1) becomes, using (2),

$$\sum_n N_n n^2S^2=NS^2\{p+a_2p^2+a_3p^3+\ldots\}, \tag{4}$$

and using (3), together with the relation $\sum_n nN_n=Np$,

$$\tfrac{1}{3}\sum N_n(nS+n^2S^2)=NS^2\left\{\frac{S+1}{S}p+a_2p^2+a_3p^3\ldots\right\}. \tag{5}$$

The a's depend only on the topology of the lattice and are the same in (4) and (5). Thus χ diverges at the same value of p in both models. The $N_{n,\alpha}$ and the a_n have been evaluated up to and including $n=5$. The critical concentration p_c has been evaluated by estimating the radius of convergence from the sequence $|a_n|^{-1/n}$ and by inverting the series. The inverted series was then truncated after 2, 3, 4, and 5 terms and equated to zero to obtain a sequence of approximations. The resulting values of p_c are as follows and the convergence of the method leads us to believe that they are accurate to within 10%. Simple cubic: 0.28; body-centered cubic: 0.22; face-centered cubic: 0.18; plane square: 0.48; plane honeycomb: 0.49; triangular net: 0.36. The finite value of p_c predicted for plane lattices in the Heisenberg model is at first sight in contradiction with the spin wave theory which predicts no ordering; although they can be reconciled if $T_c=0$, $p_c\leq p\leq 1$. Series methods,[1] however, seem to indicate a finite T_c for these lattices, so that unexplained discrepancies remain.

These results may be compared with those obtained by other methods. Brout[2] makes a more sophisticated attack on the problem but the simplest approximation which he actually evaluates is only equivalent to considering the first two terms of the series. Elliott[3] has evaluated p_c in the constant-coupling approximation, but this

also places undue emphasis on the first two terms in the series[4] (5) and leads to a dependence of p_c on S which is a spurious result of the approximation.

The above theorem holds if more than next nearest neighbor interactions are included provided they are all ferromagnetic. It does not hold in antiferromagnetics[5] where the particular case of MnF_2 and CoF_2 mixed with ZnF_2 has been considered for comparison with the experimental results of Baker and Lourens.[6] The method of series expansion in p may be extended to finite T to obtain the variation of T_c with p.[4,5]

[1]G. S. Rushbrooke and P. J. Wood, Proc. Phys. Soc. (London) A68, 1161 (1955).
[2]R. Brout, Phys. Rev. 115, 824 (1959).
[3]R. J. Elliott, J. Phys. Chem. Solids (to be published).
[4]D. J. Morgan and G. S. Rushbrooke (to be published).
[5]R. J. Elliott and B. R. Heap (to be published).
[6]J. M. Baker and J. A. Lourens (to be published).

INTERACTION OF HIGH-ENERGY PHONONS IN GERMANIUM*

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Many authors have investigated the non-Ohmic conductivity of germanium.[1-6] The theoretical explanation advanced by Shockley[7] to explain the experimental results of Ryder and others was that electrons which have been very much accelerated by the applied electric field will emit phonons and lose an appreciable fraction of their energy. Recently there have been attempts to observe these high-energy phonons. Aigrain[8] tried to detect them by means of the momentum they would transfer to electrons in a region of the sample where the electrons are "cold"; Hubner and Shockley[9] succeeded in an experiment similar to that attempted by Aigrain and observed such a transmitted phonon drag in silicon. We will report an experiment in which a transport of energy by phonons excited by hot electrons is observed.

In our experiment, a short voltage pulse produced low-temperature breakdown in the portion AB of a very pure n-type antimony-doped germanium crystal ($N_D - N_A \sim 4\times10^{13}$ cm^{-3}, $N_A \sim 5\times10^{11}$ cm^{-3}) placed outside an S-band microwave cavity (Fig. 1). A dc voltage supplied by a battery could be applied to the portion BC of the sample that was in the microwave cavity. The sample had its axis along the [111] direction.

At 4.2°K when the germanium sample was at thermal equilibrium with the liquid helium surrounding it, the unloaded Q of the cavity was of the order of 4000. Losses were principally due to the metal walls. (Losses originating from the sample would result in a value of Q of about 10^6.)

The cavity was placed on one of the symmetrical arms of a hybrid Tee; on the other there was a load with an impedance selected so that no power fed into the H arm could be detected out of the E arm. The detector used was a superheterodyne system. By means of such an arrangement, small changes in the properties of the cavity can be easily detected: Weber[10] was able to measure frequency shifts of 10^{-5}%; and we could easily

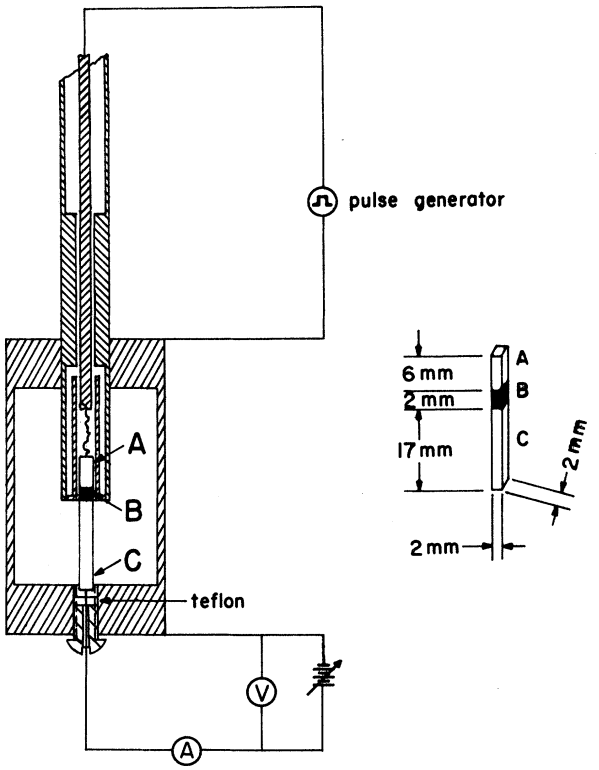

FIG. 1. Experimental arrangement for producing and detecting the "phonon beam." The portion of the sample in the vicinity of B is covered with tin. The cross section of the sample is 2×2 mm^2; all the contacts are made with tin.