THEORY OF A HEISENBERG FERROMAGNET IN THE RANDOM PHASE APPROXIMATION*

F. Englert†

Laboratory of Atomic and Solid State Physics and Department of Physics, Cornell University, Ithaca, New York

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The aim of this Letter is to present a solution of the ferromagnetic problem in the Heisenberg model in random phase approximation (RPA). This is conjectured to be the high-density limit theory. In this preliminary account we restrict ourselves to spin one-half and to a study of magnetic properties alone.

The principal result of the analysis is a magnetization curve which imitates conventional spin wave theory very closely but with the existence of renormalized spin wave frequencies according to $\omega(\vec{q};T)=\omega(\vec{q};0)M(T)/M(0)$, where $M(T)$ is the magnetization at temperature T, in accordance with an idea of Brout.[1] The theory goes over to the spherical model[2] at temperature higher than the Curie temperature for small static fields with the Curie point given by the spherical model value. Thus we have a consistent extrapolation formula through the whole temperature range possessing rigorously correct limiting properties for $T\to 0$ and $T\to\infty$, as well as a presumably accurate value of the Curie temperature in the limit of high density.

We write the Heisenberg Hamiltonian in the form

$$\mathcal{H}=-2\sum_{\vec{q}}v(\vec{q})\{S^z(\vec{q})S^z(-\vec{q})+\tfrac{1}{2}[S^+(\vec{q})S^-(-\vec{q})+S^-(\vec{q})S^+(-\vec{q})]\},\quad(1)$$

where we have Fourier-analyzed the exchange potential $v(\vec{r})$ and the spin operators $S(\vec{q})=N^{-1/2}\sum_i S_i\exp(i\vec{q}\cdot\vec{r}_i)$. The random phase approximation can be summarized by the simplified commutation rule which expresses the decoupling of different Fourier components:

$$[S^+(\vec{q}),S^-(\vec{q}\,')]_-=\delta_{\vec{q},-\vec{q}\,'}(2/\sqrt{N})S^z(0),$$
$$[S^z(\vec{q}),S^\pm(\vec{q}\,')]_-=\pm\delta_{\vec{q},0}(1/\sqrt{N})S^\pm(\vec{q}\,'),\quad(2)$$

where N is the number of spins.

We apply to the system a finite longitudinal magnetic field H and switch on adiabatically an infinitesimal transverse rotating magnetic field of wave vector $\vec{q}\,'$. This adds to the Hamiltonian (1) a term,

$$\mathcal{H}'=-S^+(-\vec{q}\,')h_{\vec{q}\,'}^-e^{i(\omega-i\epsilon)t}-g\mu\sum_i S_i^zH.\quad(3)$$

Denoting by $\langle O\rangle_{T,H}$ the thermal average at temperature T and external magnetic field H of the operator O, we may define the adiabatic spin susceptibility,

$$\chi_{\vec{q}\,'}^-(\omega)_{T,H}=\lim_{h_{\vec{q}\,'}^-\to 0}\left(\frac{\langle S^-(\vec{q}\,')\rangle_{T,H}}{h_{\vec{q}\,'}^-e^{i(\omega-i\epsilon)t}}\right).\quad(4)$$

Writing the Heisenberg equations of motion, one easily finds with help of (2),

$$\chi_{\vec{q}\,'}^-(\omega)_{T,H}=\frac{R}{[v(0)-v(\vec{q})]2R+g\mu H-\hbar\omega+i\epsilon},\quad(5)$$

where R is the reduced magnetization, $R=2N^{-1}\times\langle\sum_i S_i^z\rangle_{T,H}$. From the Kubo formalism,[3] the appropriate form of the fluctuation-dissipation theorem is easily derived[4]:

$$\int_{-\infty}^{\infty}\frac{1}{1-\exp(\hbar\omega/kT)}\,\mathrm{Im}\chi_{\vec{q}}^-(\omega)_{T,H}d\omega=(i\pi/\hbar)\langle S^-(\vec{q})S^+(-\vec{q})\rangle_{T,H},\quad(6)$$

where χ in Eq. (6) is the adiabatic susceptibility, which in RPA is given by Eq. (5). Substituting Eq. (5) into Eq. (6), we have

$$\frac{R}{\exp\{[(v(0)-v(\vec{q}))2R+g\mu H]/kT\}-1}=\langle S^-(\vec{q})S^+(-\vec{q})\rangle_{T,H}.\quad(7)$$

In a completely similar way, we obtain

$$\frac{R}{1-\exp\{-[(v(0)-v(\vec{q}))2R+g\mu H]/kT\}}=\langle S^+(\vec{q})S^-(-\vec{q})\rangle_{T,H}.\quad(8)$$

Using the sum rule,

$$\sum_{\vec{q}}[S^+(\vec{q})S^-(-\vec{q})+S^-(\vec{q})S^+(-\vec{q})]=N,\quad(9)$$

we obtain the following equation for the magnetization:

$$N = R\sum_{\vec{q}} \coth\left(\frac{[v(0) - v(\vec{q})]2R + g\mu H}{2kT}\right). \qquad (10)$$

Equation (10) has two classes of solutions in the limit $H \to 0$.

(a) $H \to 0$, $R \nrightarrow 0$, or the ferromagnetic region.

The physical interpretation of (10) in the ferromagnetic region is best seen by writing (10) (as $H \to 0$) in "spin wave" form:

$$n = \sum_{\vec{q}} \frac{R}{\exp\{[v(0) - v(\vec{q})]2R/kT\} - 1}, \qquad (11)$$

where n is the number of "spin deviations"; $n = \frac{1}{2}N(1 - R)$. Thus the magnetization curve (11) appears as a consequence of the existence of collective spin motions generalizing at finite temperature the well-known spin waves by renormalizing their frequency by a factor R. This concept of renormalized spin waves, which leads to a T^3 correction to the usual spin-wave theory for the magnetization curve at low temperature, seems to be in contradiction with Dyson's theory,[5] and this discrepancy will be studied in detail in the future. This concept and the T^3 law which results has been already introduced by Brout and Haken[1] from a physical point of view. The magnetization curve obtained in reference 1, however, is not the same as the present one. A detailed study has been made of this discrepancy by Brout. It has turned out that the reasoning in reference 1 contained an inconsistency in the evaluation of traces in RPA. In a subsequent publication it will be shown how the final result [Eq. (10)] can be obtained by a consistent RPA in the partition function.

The Curie point determined from (10) or (11) is obtained by taking the limit $R \to 0$, $H \to 0$:

$$1/kT_c = F(1)/v(0), \qquad (12)$$

with

$$F(1) = \frac{1}{N}\sum_{\vec{q}} \frac{1}{1 - [v(\vec{q})/v(0)]},$$

which is the Curie point of the spherical model.[2]

(b) $H \to 0$, $R \to 0$, or the paramagnetic region.

The susceptibility being defined by $\chi = \frac{1}{2}g\mu R/H$ as $H \to 0$, one finds immediately from (10)

$$1 = \frac{1}{N}\sum_{\vec{q}} \frac{1}{\beta[v(0) - v(\vec{q})] + \beta(g\mu/2)^2/\chi}. \qquad (13)$$

If we write

$$\chi = \left(\frac{g\mu}{2}\right)^2 \beta \frac{1}{1 - \beta[v(0) - \delta]}, \qquad (14)$$

one sees from (13) that δ is determined by

$$1 = \frac{1}{N}\sum_{\vec{q}} \frac{1}{1 - \beta[v(\vec{q}) - \delta]}, \qquad (15)$$

which shows the equivalence of our paramagnetic curve with that obtained from the spherical model.[2] Because of the work of Brout,[6] we speculate on the present theory as the high-density limit theory.

A complete discussion of these results, as well as the thermodynamic and nonequilibrium properties, will be published in a forthcoming paper.

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†On leave of absence from Université Libre de Bruxelles, Bruxelles, Belgium.

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