a phase proportional to the winding number of the particle trajectories around one another. This phase can be regarded as an interaction of a peculiar type (long range, dependent only on topology of paths); or we may incorporate it into the states. In the latter procedure, the amplitude for two anyons at positions $\vec{r}_1(t_0)$, $\vec{r}_2(t_0)$ to propagate along given paths to positions $\vec{r}_1(t_1)$, $\vec{r}_2(t_1)$ depends on the angle through which the relative position $\vec{r}_1 - \vec{r}_2$ has turned. The phase is not in general unity for a 2π turn, and we must allow states as in (5) or (6)—not 2π periodic in the angle—to keep track of it. The mathematical analogy of all this to winding numbers and θ vacuums in gauge theories[9] is very close.

The situation for three or more anyons seems very complicated. The configuration space for three identical anyons will be $C = \mathfrak{R}^2 \times \mathfrak{R}^2 \times \mathfrak{R}^2/S - I$ with symmetrical points identified [e.g., $(\vec{r}_1, \vec{r}_2, \vec{r}_3) \sim (\vec{r}_2, \vec{r}_1, \vec{r}_3)$] and identical points [e.g., $(\vec{r}_1, \vec{r}_2 = \vec{r}_1, \vec{r}_3)$] excluded. The wave function is defined on the universal covering of this space, with conditions like (6) for points in the covering space which project to one point in C. The universal covering space seems very awkward to parametrize and I have not made much progress with it. It is certainly an intriguing mathematical problem to see how the statistical mechanics of many free anyons interpolates between bosons and fermions.

I am very grateful to Sidney Coleman for pressing me to think through this subject and for several helpful suggestions. This research was supported by the National Science Foundation through Grant No. PHY77-27084.

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Equivalence of the Two-Dimensional Directed-Site Animal Problem to Baxter's Hard-Square Lattice-Gas Model

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(Received 7 June 1982)

The directed-site animal problem on the square and triangular lattices is shown to be equivalent to Baxter's hard-square lattice-gas model with anisotropic next-nearest-neighbor interactions at a disorder point. The exact solution of the latter is used to determine the animal numbers as functions of their size in the two cases. Expressions in closed form are proposed for the number of animals on an infinite strip in terms of width and animal size, and for the average transverse extent of animals of a given size.

PACS numbers: 05.50.+q, 05.70.Jk, 64.60.Cn

The directed animal problem,[1,2] which is related to the problem of directed percolation,[3] has been studied recently using series expansions,[1] in the Flory approximation,[4] using the ϵ expansion,[5] and with finite-size scaling techniques.[6] The problem in d dimensions is related to the Lee-Yang edge-singularity problem in $d-1$ dimensions.[7] In this Letter the $d=2$ problem is shown to be equivalent to a (square-)lattice-gas model with nearest-neighbor (NN) exclusion and anisotropic next-nearest-neighbor (NNN) interactions studied earlier by Baxter and collaborators.[8] This model has an anisotropic, but reflection-symmetric, Hamiltonian. It can be solved along a line in the three-dimensional interaction-parameter space, and the expression for the density of the lattice gas along this line establishes the validity of the expressions for the number of directed animals on the square and triangular lattices conjectured earlier.[2]

Consider an $N \times N$ square lattice with periodic boundary conditions and associate a variable $n_{i,j}$ (taking values 0 and 1) with each site (i,j). The Hamiltonian of the system is given by

$$H_A = J\sum_{ij} n_{i,j}(1-n_{i,j-1})(1-n_{i-1,j}) - \mu \sum n_{i,j}. \tag{1}$$

The corresponding partition function is defined by

$$Z_A(\alpha = \exp(-J), z = \exp(\mu), N) = \operatorname{Tr}\exp(-H_A). \tag{2}$$

The free energy per site in the thermodynamic limit will be denoted by $f(\alpha, z)$. In the limit $J \to +\infty$, configurations containing a site (i,j) such that $n_{i,j}$ is 1, and both $n_{i-1,j}$ and $n_{i,j-1}$ are zero, have infinite energy, and do not contribute to Z_A. This is the directed-animals constraint. The constraint also implies that there are no sites distinguishable as sources, and that all animals must wind around the torus. The smallest animals are of size N, and there are only $2N$ such distinct configurations. This implies that

$$Z_A(\alpha = 0, z, N) = 1 + 2Nz^N + \text{higher-order terms in } z, \tag{3}$$

since the coefficient of z^s in Eq. (3) is the number of animals of size s on the $N \times N$ torus. This coefficient increases at most as λ^s, where λ is a constant independent of N. Note that disconnected animals (which are allowed in the sum over states) only contribute to the coefficients of z^{2N} and higher powers. In the thermodynamic limit $N \to \infty$, the right-hand side of Eq. (3) converges to 1 for $|z| < \lambda^{-1}$, and hence

$$f(\alpha = 0, z) = 0, \quad \text{for } |z| < \lambda^{-1}. \tag{4}$$

Thus, for $\alpha = 0$ and $|z| < \lambda^{-1}$, in the thermodynamic state of the Hamiltonian H_A, all sites are unoccupied. First-order perturbation theory about this unoccupied state shows that

$$-\frac{\partial}{\partial \alpha} f(\alpha, z)\Big|_{\alpha=0} = \sum_{s=1}^{\infty} A_s z^s \equiv A(z), \quad \text{for } |z| < \lambda^{-1}. \tag{5}$$

Here A_s is the number of directed animals of size s growing in an infinite plane with a single point source at the origin. For large s, A_s varies as λ^s with $\lambda = 3$.[6] $A(z)$ is the generating function for the directed-site animal problem on the square lattice, and is conjectured to be given by[2]

$$A(z) = [(1+z)^{1/2}(1-3z)^{-1/2} - 1]/2. \tag{6}$$

Define $t_{i,j} = n_{i,j}(1-n_{i,j-1})(1-n_{i-1,j})$, and use $\exp(-Jt_{i,j}) = \{1-(1-\alpha)t_{ij}\}$ in Eq. (2) to get

$$Z_A(\alpha, z, N) = \operatorname{Tr}\prod_{i,j}[\{1-(1-\alpha)t_{i,j}\}z^{n_{i,j}}].$$

In the expansion of Z_A into a sum of products of $t_{i,j}$'s, for each term the summation over $n_{i,j}$'s can be done explicitly. The result can be expressed as the statistical weight of a configuration of a lattice gas with NN exclusion, whose occupation numbers $\tilde{n}_{i,j}$'s are the powers of $t_{i,j}$'s in the term, and the Hamiltonian is given by

$$H_B = \sum_{i,j}(\tilde{L}\tilde{n}_{i,j}\tilde{n}_{i-1,j+1} + \tilde{M}\tilde{n}_{i,j}\tilde{n}_{i+1,j-1} - \tilde{\mu}\tilde{n}_{i,j}); \tag{7}$$

where

$$\tilde{M} = 0; \quad e^{-\tilde{L}} = 1 + z; \quad e^{\tilde{\mu}} = -(1-\alpha)z(1+z)^{-3}. \tag{8}$$

H_B is the Hamiltonian of the Baxter model with NNN interactions $\tilde{L}$ and $\tilde{M}$, and chemical potential $\tilde{\mu}$. The directed-animal problem $\alpha = 0$, $z > 0$ corresponds to the Baxter gas with negative activity $(e^{\mu} < 0)$ and attractive interactions. If $g(e^{\tilde{\mu}}, e^{-\tilde{L}}, e^{-\tilde{M}})$ is the Gibbs free energy per site for the Hamiltonian H_B, we get

$$g(e^{\tilde{\mu}}, e^{-\tilde{L}}, 1) = \ln(1+z) + f(\alpha, z). \tag{9}$$

In spin language, H_B with $M = 0$ is the Hamiltonian of an antiferromagnetic Ising model with anisotropic NN couplings in an external field defined on a triangular lattice. This problem can be solved exactly[9] for special values of coupling constants called disorder points.[10] It has been shown by Baxter[11] that Eqs. (7) and (8) with $\alpha = 0$ correspond to Verhagen's parametrization

$$b = 1, \quad z = (a-1)/(1-a+a^2). \tag{10}$$

Verhagen's solution is valid for $0 \leq a \leq 1$, which corresponds to $-1 \leq z \leq 0$, or equivalently $\tilde{L} \geq 0$. By analytic continuation, the solution can be extended to the regime $-1 \leq z \leq \frac{1}{3}$. Substituting Eqs. (8) and (10) in Eq. (33) of Ref. 9, we get the density of the lattice gas,

$$\rho(e^{\tilde{\mu}} = e^{3\tilde{L}} - e^{2\tilde{L}}, e^{-\tilde{L}}, 1) = [1 - (4e^{\tilde{L}} - 3)^{-1/2}]/2. \tag{11}$$

Differentiating Eq. (9) with respect to α, and substituting $e^{-\tilde{L}}=1+z$, we can easily see that Eq. (11) establishes Eq. (6) stated earlier as a conjecture. It may be noted here that the line $e^{\tilde{\mu}}+e^{2L}-e^{3L}=\tilde{M}=0$ does not satisfy Baxter's factorizability condition,[8] except in the zero-density limit.

A similar treatment works for the directed site animals on a triangular lattice with the definition

$$t_{i,j}=n_{i,j}(1-n_{i,j-1})(1-n_{i-1,j})(1-n_{i-1,j-1}).$$

The parameters of the corresponding lattice gas are given by

$$M=\infty;\quad e^{-L}=1+z;\quad e^{\tilde{\mu}}=-(1-\alpha)z(1+z)^{-4}. \tag{12}$$

Baxter[11] has shown that this case also corresponds to a disorder point of the Hamiltonian H_B, and the largest eigenvector of the corresponding transfer matrix can be obtained by a generalization of Verhagen's *Ansatz*. The analogs of Eqs. (4) and (11) are

$$g(e^{\tilde{\mu}}=e^{4\tilde{L}}-e^{3\tilde{L}},\, e^{-\tilde{L}},\, 0)=-\tilde{L} \tag{13}$$

and

$$\rho(e^{\tilde{\mu}}=e^{4\tilde{L}}-e^{3\tilde{L}}, e^{-\tilde{L}}, 0)=[1-(5-4e^{-\tilde{L}})^{-1/2}]/2. \tag{14}$$

Putting $e^{-\tilde{L}}=1+z$ in the above, we get the generating function for the triangular-lattice directed-site animals, proving Eq. (16) of Ref. 2.

The lattice-gas equivalent of the directed-animal problem on the hexagonal lattice can be obtained in an analogous fashion. Let x_1 and x_2 be the weights of occupied sites on the two sublattices of the hexagonal lattice in the directed-animal problem. The animal generating function $G_2^{\text{hex}}(x_1,x_2)$ is defined to be the sum of weights of all animals growing from a point source at a site with two bonds directed outwards. $G_2^{\text{hex}}(x_1, x_2)$ is related to the generating function for the cluster size distributions $G^{\text{sq}}(x,y)$ of the directed percolation problem (in the notation of Ref. 2) by the formula

$$G_2^{\text{hex}}(x_1,x_2)=G^{\text{sq}}(x_1x_2,\,1+x_1)/x_1.$$

The lattice-gas equivalent of the directed-animals problem on the hexagonal lattice with staggered activities gives a Hamiltonian formulation of the directed *percolation* problem on the square lattice.

The equivalence between the directed-animals problem and the Baxter gas is valid even if the interaction parameters are site dependent. This implies that the correlation functions of the two problems are related, and, as in the animal problem, the lattice-gas model must show two correlation-length exponents $\nu_\parallel$ and $\nu_\perp$ near the critical point. This is unlike the usual behavior of undirected models with anisotropy where there is a single direction-independent exponent ν, and is due to the fact that the lattice gas has competing interactions and is at a disorder point.[10] The Baxter Hamiltonian with $e^{\tilde{\mu}}>0$ and factorizability condition shows a divergence of the correlation length with an isotropic exponent $\nu=\frac{5}{6}$, except at some special points. This differs significantly from the estimated values $\nu_\parallel=0.818\pm0.001$ and $\nu_\perp=\frac{1}{2}$ for the directed animals. Clearly, the "unphysical" region of the Baxter model with negative activities is of much interest.

The exponent $\nu_\perp$ for the directed animals in two dimensions has been shown[7] to be $\frac{1}{2}$. Consider directed-site animals growing from a point source (say at the origin) on a square lattice and constrained to lie completely between the lines $y\leqslant x+d_1$ and $y\geqslant x-d_2$ ($d_1, d_2\geqslant 0$). Let $A(s,d_1,d_2)$ be the number of such animals of size s. Results of explicit enumeration suggest that $A(s,d_1,d_2)$ is exactly given by the formula

$$A(s,d_1,d_2)=(d_1+d_2+3)^{-1}\sum_{m=1}^{d_1+d_2+2}(1+2\cos m\delta)^{s-1}(1+e^{im\delta})[1-\exp(im\delta d_1+im\delta)],$$

where $\delta=2\pi/(d_1+d_2+3)$. This formula has been verified for all $0\leqslant d_1, d_2\leqslant 8$, and $s\leqslant 12$, and is expected to hold for all s, d_1, and d_2. The special case $d_1=0$, $d_2=\infty$ corresponds to animals growing in the octant $0\leqslant y\leqslant x$. In this case $A(s,0,\infty)$ varies as $3^s s^{-3/2}$ for large s (compare to $3^s s^{-1/2}$ for unconstrained directed animals).

With use of the explicit expression for $A(s,d_1,d_2)$, the fractional number of animals having a width $\xi_\perp$ [defined here as the minimum value of (d_1+d_2+1) such that the animal lies completely between the lines $y\leqslant x+d_1$ and $y\geqslant x-d_2$] is easily calculated. Averaging over all animals of size s gives

$$\langle\xi_\perp\rangle=2\times3^{s-1}A_s^{-1}-1.$$

For large s, $\langle\xi_\perp\rangle$ varies as $s^{1/2}$, consistent with $\nu_\perp=\frac{1}{2}$.

I would like to thank Professor R. J. Baxter for a very helpful correspondence, in particular for clarifying the question of the range of validity of Eq. (11), and for permission to quote his unpublished results [Eqs. (10), (11), and (14)]. I would also like to thank Dr. M. Barma, Dr. M. K. Phani, and Dr. A. K. Raina for critically reading the manuscript.

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