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LETTERS

Quantum Mechanics of Fractional-Spin Particles

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Composites formed from charged particles and vortices in (2 + 1)-dimensional models, or flux tubes in three-dimensional models, can have any (fractional) angular momentum. The statistics of these objects, like their spin, interpolates continuously between the usual boson and fermion cases. How this works for two-particle quantum mechanics is discussed here.

In a recent note[1] I showed that charged particles orbiting around magnetic flux tubes have orbital angular momentum integer $+q\Phi/2\pi$; this phenomenon is realized for example in the vortices of a type-II superconductor and in string solutions of gauge theories.[1] Closely related observations were made previously by Hasenfratz,[2] and recently by Goldin and Simon.[3] See also the discussion by Peshkin.[4] If there is a generalized spin-statistics connection, we must expect that the flux-tube–particle composites have unusual statistics, interpolating between bosons and fermions. Since interchange of two of these particles can give *any* phase, I will call them generically anyons.

In this paper some elementary examples in the quantum mechanics of anyons are worked out. Description of these particles requires some widening of the notion of a wave function. Also, we will see that the energy levels of a system of two noninteracting anyons are not in general simply related to the one-anyon levels.

Although practical applications of these phenomena seem remote, I think they have considerable methodological interest and do shed light on the fundamental spin-statistics connection. Some related work has been done previously.[2,5,6]

One anyon.—Let us recall how the fractional L_z arises. Charged particles orbiting around a flux tube carrying flux Φ are subject to an azimuthal vector potential

$$A_\varphi = \Phi/2\pi r. \tag{1}$$

Although the potential gives vanishing magnetic field strength, and therefore is negligible in classical physics, it does play a role in quantum mechanics.[7] It is convenient to eliminate A_φ by a gauge transformation:

$$A_i' = A_i - \partial_i \Lambda = 0, \quad \Lambda = \Phi\varphi/2\pi. \tag{2}$$

The required Λ is, however, not a well-defined (2π periodic) function of the angle φ. This reflects itself in the transformation of charged-particle wave functions:

$$\psi'(\varphi) = e^{iq\Phi\varphi/2\pi}\psi(\varphi). \tag{3}$$

In fact, since $\psi(\varphi)$ is 2π periodic we find from (3) that

$$\psi'(\varphi + 2\pi) = e^{iq\Phi}\psi'(\varphi). \tag{4}$$

The allowed angular wave functions $\psi'(\varphi) \sim e^{im\varphi}$ therefore have $m = \text{integer} + q\Phi/2\pi$, which is to be interpreted as the spectrum of orbital angular momenta. In the primed system we have arranged

for the charged particle to obey the free Schrö-dinger equation (i.e., with no vector potential) at the cost of introducing unusual boundary conditions.

The boundary conditions (4) may be incorporated systematically by formulating quantum mechanics on sections or fiber bundles,[8] or pragmatically by restricting the Hilbert space of angular wave functions as above.

Two anyons, angular momentum.—Now consider two identical anyons, e.g., tightly bound flux-tube–charged-particle composites. Suppose that the electrostatic forces are small and can be treated as a perturbation (that is, consider the limit $q \to 0$, $q\Phi$ fixed). Each anyon is a charged particle, and its motion around the other is analyzed as in the previous section. If we express the two-body wave function Ψ in terms of the center-of-mass coordinates R, θ and the relative coordinates r, φ then the condition analogous to (4) is

$$\Psi(R,\ \theta;\ r,\ \varphi + 2\pi) = e^{4\pi i \Delta}\ \Psi(R,\ \theta;\ r,\ \varphi), \qquad (5)$$

where $\Delta \equiv q\Phi/2\pi$ is the anomalous angular momentum.

A more stringent condition holds if the anyons are identical, for then rotation of φ through π amounts to an interchange of particles and must give a definite phase. To analyze this, we must go back to the original underlying theory. Let us suppose for definiteness that all the fields describing the flux tube and the charged particle are bosonic. Then if one simply superposes the fields, e.g., adding the gauge fields and using product wave functions for the charged particles, then the requirement of Bose statistics is that the total wave function be unchanged by the interchange $\varphi \to \varphi + \pi$. The superposition can only be done unambiguously in the regular gauge (that is, the original unprimed gauge above), where the wave functions are single valued. However, this gauge has the disadvantage that it introduces peculiar long-range interactions between the anyons. In fact, if one anyon moves it will interact with the potential of the other through the

$q\vec{\mathrm{v}}\cdot\vec{\mathrm{A}}$ term in the charged-particle Lagrangian. We can eliminate these long-range interactions by going to a singular gauge as before, again at the cost of introducing unusual boundary conditions. It is straightforward to see that the resulting condition is

$$\Psi(R,\ \theta;\ r,\ \varphi + \pi) = e^{2\pi i \Delta}\ \Psi(R,\ \theta;\ r,\ \varphi). \qquad (6)$$

Thus for $\Delta = 0$ we have effectively bosons, for $\Delta = \frac{1}{2}$ fermions, with continuous interpolation between these cases.

If the underlying charged-particle fields were fermions, there would of course be an additional minus sign in (6). Since the spin of the anyon is $s_a = \text{integer} + \Delta + s$, where s is the intrinsic spin of the charged particle, the phase accompanying an interchange is $\exp(2\pi i s_a)$, as we would expect on the basis of a generalized spin-statistics connection.

If the angular wave function in (6) goes like $e^{im\varphi}$, then

$$m = (\text{even integer}) + 2\Delta. \qquad (7)$$

There is a centrifugal barrier when $\Delta \neq 0$, since the relative angular momentum cannot vanish.

Two anyons, harmonic well.—In general the condition (6) makes it very difficult to solve problems with two anyons in an external potential. In general the Schrödinger equation does not separate in the r, φ variables and imposition of (6) greatly complicates its solution.

A harmonic potential, on the other hand, does separate and can be analyzed completely. Let us normalize so that the allowed one-anyon energy levels are ϵ, 2ϵ, 3ϵ, ... which have degeneracies 1, 2, 3, ... for a two-dimensional harmonic oscillator (and integral angular momentum!). The two-anyon Hamiltonian in this harmonic well can be separated into two oscillators—the center-of-mass $(R,\ \theta)$ oscillator and the relative motion $(r,\ \varphi)$ oscillator. For the latter, only the angular momenta (7) are allowed. As a result the (energies, degeneracies) are, for the c.m. oscillator,

$$(\epsilon, 1),\ (2\epsilon, 2),\ (3\epsilon, 3), \ldots, \qquad (8)$$

and for the relative oscillator,

$$[(1+2\Delta)\epsilon, 1],\ [(3+2\Delta)\epsilon, 2],\ [(5+2\Delta)\epsilon, 3], \ldots;\ \ [(3-2\Delta)\epsilon, 1],\ [(5-2\Delta)\epsilon, 2],\ [(7-2\Delta)\epsilon, 3], \ldots. \qquad (9)$$

The total energies and degeneracies may of course be read off from these. A notable qualitative feature is that when $\Delta \neq 0, \frac{1}{2}$ the two-anyon energy levels are not simply related to the one-anyon levels!

Comments.—It is illuminating to think about the

need for extended wave functions—e.g., not necessarily 2π periodic in φ—in terms of path integrals. Consider first, as above, two anyons. The contribution of a given path (a pair of particle trajectories) to the transition amplitude has

a phase proportional to the winding number of the particle trajectories around one another. This phase can be regarded as an interaction of a peculiar type (long range, dependent only on topology of paths); or we may incorporate it into the states. In the latter procedure, the amplitude for two anyons at positions $\vec{r}_1(t_0)$, $\vec{r}_2(t_0)$ to propagate along given paths to positions $\vec{r}_1(t_1)$, $\vec{r}_2(t_1)$ depends on the angle through which the relative position $\vec{r}_1 - \vec{r}_2$ has turned. The phase is not in general unity for a 2π turn, and we must allow states as in (5) or (6)—not 2π periodic in the angle—to keep track of it. The mathematical analogy of all this to winding numbers and θ vacuums in gauge theories[9] is very close.

The situation for three or more anyons seems very complicated. The configuration space for three identical anyons will be $C = \Re^2 \times \Re^2 \times \Re^2/S - I$ with symmetrical points identified [e.g., $(\vec{r}_1, \vec{r}_2, \vec{r}_3)$ $\sim (\vec{r}_2, \vec{r}_1, \vec{r}_3)$] and identical points [e.g., $(\vec{r}_1, \vec{r}_2 = \vec{r}_1, \vec{r}_3)$] excluded. The wave function is defined on the universal covering of this space, with conditions like (6) for points in the covering space which project to one point in C. The universal covering space seems very awkward to parametrize and I have not made much progress with it. It is certainly an intriguing mathematical

problem to see how the statistical mechanics of many free anyons interpolates between bosons and fermions.

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Equivalence of the Two-Dimensional Directed-Site Animal Problem to Baxter's Hard-Square Lattice-Gas Model

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The directed-site animal problem on the square and triangular lattices is shown to be equivalent to Baxter's hard-square lattice-gas model with anisotropic next-nearest-neighbor interactions at a disorder point. The exact solution of the latter is used to determine the animal numbers as functions of their size in the two cases. Expressions in closed form are proposed for the number of animals on an infinite strip in terms of width and animal size, and for the average transverse extent of animals of a given size.

The directed animal problem,[1,2] which is related to the problem of directed percolation,[3] has been studied recently using series expansions,[1] in the Flory approximation,[4] using the ϵ expansion,[5] and with finite-size scaling techniques.[6] The problem in d dimensions is related to the Lee-Yang edge-singularity problem in $d - 1$ dimensions.[7] In this Letter the $d = 2$ prob-

lem is shown to be equivalent to a (square-)lattice-gas model with nearest-neighbor (NN) exclusion and anisotropic next-nearest-neighbor (NNN) interactions studied earlier by Baxter and collaborators.[8] This model has an anisotropic, but reflection-symmetric, Hamiltonian. It can be solved along a line in the three-dimensional interaction-parameter space, and the expression