Electromagnetism, Spin, and Statistics

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Delicacies in quantum physics for magnetic sources carrying electric charge, recently exposed by Wilczek, are studied further, with emphasis on the meaning of angular momentum for these systems, and the nature of phases which affect the statistics of indistinguishable objects. Wilczek has uncovered a remarkable two-body interaction which leads to a many-body system in two dimensions changing from Bose gas to Fermi gas as the interaction parameter is varied. A nontrivial realization of these notions might occur in a thin sheet of superconductor with many magnetic vortices.

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Wilczek recently has explored subtle quantum properties of a system containing electric charge and a magnetic source which is rotationally symmetric in two[1] or in three[2] space dimensions. The latter work[2] confirms and builds on Witten's remarkable discovery[3] that a unit magnetic monopole becomes a dyon[4] with the irrational electric charge $q = e\theta/2\pi$ if the vacuum is characterized by chiral angle θ, where e is the unit of charge.

Jackiw,[5] noting that angular momentum in two dimensions is determined only up to an additive constant, argued for a definition of spin different from Wilczek's.[1] There follows below an examination of the aesthetics of different definitions both for spin and for statistics, leading to the observation that Wilczek has introduced an idealized two-body interaction which implies many-body dynamics completely determined by a single real parameter. As the parameter varies from one extreme to the other of its range, the system changes from a free Bose gas to a free Fermi gas. In these limiting cases, Wilczek's definition of spin is required to maintain the familiar connection of spin and statistics. For intermediate parameters the best interpretation depends on the yet unknown solution of the many-body problem.

At one nontrivial but solvable extreme, there might be a physical realization in which a thin, uniform superconducting sheet supports a Bose condensation of objects each consisting of a vortex with an electron bound to it. In three dimensions, where spin and statistics should be unambiguous and connected in a well-known way, several different approaches confirm Wilczek's statement[2] that the spin-statistics connection works also for Witten dyons.

Wilczek[1] discusses an object consisting of an infinitely long, cylindrically symmetric, ideal impenetrable solenoid to which is bound an electrically charged particle circulating around it. Let us call this object a "cyon" for short. Note that the infinite solenoid length implies infinite cyon mass, so that quantum interference effects in cyon-cyon interactions are unobservable in principle. Nevertheless, one may abstract from this three-dimensional system a model for objects moving in two space dimensions, in which case the cyons could be taken to have finite mass. We shall be interested in the spin and statistics of cyons, and begin by seeking the appropriate spin or angular momentum operator.

For this purpose Wilczek chooses what may be called the kinetic angular momentum $\vec{M} = \vec{r} \times [\vec{p} - e\vec{A}(\vec{r})]$, where e is the charge, and $\vec{r}$ and $\vec{p}$ are the conjugate position and momentum coordinates of the particle. This quantity, which may also be written $\vec{r} \times m\vec{v}$, where $\vec{v}$ is the particle velocity, is a well-defined operator which has the property that for finite systems of charges and magnetic sources it is neither quantized nor conserved. In other words, it is not the full generator of rotations of the particle-magnet relative coordinates.

The actual generator, which *is* quantized, and depending on symmetries of the magnet may be partly or wholly conserved, is[5,6]

$$\vec{L} = \vec{M} + \vec{S}, \tag{1}$$

$$\vec{S} = \int d^3r\, \vec{r} \times (\vec{E} \times \vec{B})/4\pi, \tag{2}$$

where $\vec{E}$ is the electric field of the charge and $\vec{B}$ is the field of the magnet.

The physical meaning of $\vec{M}$ alone is magnetic moment; i.e., $e\vec{M}/2m$ gives the contribution to the magnetic dipole moment of the system due to a Schrödinger particle with charge e and mass m. Thus, it is certainly an interesting observable, but one would not necessarily expect it to define the spin of a charge-magnet system.

Wilczek uses the existence of $\vec{M}$ as a well-defined gauge-invariant operator to derive the Dirac quantization condition for the product of particle electric charge e with the magnetic charge g of an isolated pole. His argument may be rephrased as follows: Consider a particular component M_z, and let the origin of particle coordinates be the location of the pole. Cylindrical symmetry allows us to write the wave function as a Fourier series in $e^{im\varphi}$, where φ is azimuthal angle about the Z axis. The action of M_Z at a given polar angle χ is to multiply $e^{im\varphi}$ by $m - eg(1-\cos\chi)$ if we use the flux "north" of χ to define the vector potential, and $m + eg(1+\cos\chi)$ if we use the flux "south" of χ (these are the only two choices which maintain manifest cylindrical symmetry). Since these two choices must give the same spectrum of eigenvalues for $M_Z(\chi)$, it follows that $2eg$ must be an integer. This is an attractive way to state the result of Wu and Yang,[7] that the shift from one gauge to the other must be given by the gradient of a continuous phase factor, in this case $e^{i(m-m')\varphi}$.

The electromagnetic angular momentum $\vec{S}$ changes the result of a thought experiment proposed by Wilczek.[1] Let a solenoid be erected on the Z axis with one end at the origin of coordinates. Let an electron move upward parallel to the Z axis at a distance greater than the solenoid radius. As the electron passes the origin it suffers a shift $-e\Phi/2\pi$ in the Z component of its kinetic angular momentum $\vec{M}$. Conservation of angular momentum for the whole system implies a compensating shift $+e\Phi/2\pi$, but this shift comes in S_Z, rather than the solenoid internal angular momentum (Wilczek's guess). A similar statement applies if an electron is outside a solenoid whose flux is gradually switched on. This implementation of angular momentum conservation contrasts with the conservation of linear momentum, analyzed by Coleman and Van Vleck.[8] They showed that second-order relativistic effects cause recoil of a magnet as its current changes, just balancing the electromotive force on an electric charge outside. Both linear and angular momentum are balanced, one by mechanical motion, but the other by storage in the electromagnetic field outside the solenoid.

For the idealized cyon discussed in Ref. 1, the eigenvalues of M_Z are simply $m - e\Phi/2\pi$, where Φ is the flux enclosed by the solenoid. M_Z is conserved if and only if there are no fringing fields and the solenoid is impenetrable. Under those conditions M_Z and L_Z have the same value. However, if one imagines obtaining the cyon as a limiting configuration, starting with a solenoid of finite length and extending it, or starting with zero flux and gradually turning on a current, then L_Z is easily shown to be conserved and to have as eigenvalues only integer m. The difference between L_Z and M_Z (immaterial for infinite cyons) is due to an integrable angular momentum density spread over all space outside the cyon. If a solenoid is penetrable, then to be conserved M_Z must be supplemented by the part of S_Z due to nearby fields. This new quantity $\overline{M}_Z$ has the spectrum $m - e\Phi/2\pi$ even if the charge is inside the solenoid.

Let us turn to the issue of statistics, that is, wave-function symmetry for systems of cyons. The analysis illuminates obscure aspects in a previous demonstration of the conventional spin-statistics relation for dyons.[9] First, consider an electric charge interacting with a single ideal solenoid. Since the vector potential is curlfree, there is no magnetic force on the particle. Nevertheless, the Aharonov-Bohm effect[10] demonstrates that there is a dynamical consequence of the enclosed flux whenever it is unequal to zero modulo a quantum of flux.

Now consider the vector potential between two indistinguishable cyons. Since the electric charge in each cyon feels the vector potential of the solenoid in the other, the cyon-cyon potential is equivalent to that of a charge interacting with twice the flux in one solenoid. Therefore, it will produce dynamical effects whenever the flux in one cyon is unequal to zero modulo half a quantum of flux. The method of Ref. 9 involved removing from the two-dyon wave function a phase factor with no dynamical consequences, and simultaneously eliminating the vector potential in the relative dyon-dyon coordinates. In the present context, the only such gauge transformations allowed involve the phase factor in relative coordinates $e^{im\varphi}$. This factor changes wave-function symmetry by $(-1)^m$, and therefore could convert Bose statistics to Fermi statistics.

However, if each cyon has flux unequal to zero modulo half a quantum, then there will be an irreducible leftover dynamical vector potential, giving a nontrivial Aharonov-Bohm effect for cyon-cyon scattering, and more complex dynamics for multicyon systems. If the leftover potential and any other interactions among cyons were sufficiently small, one might expect the Bose or Fermi symmetry of a many-cyon system to be a good indication of the dynamics, leading either to a Bose condensate or to a Fermi sea as the ground state. If the interactions were stronger, the dynamics could be drastically changed. For example, clusters of cyons might behave like fermions.

Recently Wilczek[11] has made explicit implications in Ref. 1 that for intermediate fluxes in the solenoids, cyons would have intermediate statistics; that is, the phase factor on interchange of two cyons would be $e^{i\alpha}$, $\alpha = e\Phi$. He notes that this definition requires use of a path-dependent quantum mechanics in which wave functions are not single valued. I would argue that the concept of wave-function symmetry loses its usefulness under these assumptions, because we have no idea what are the implications of nonintegral α/π until we solve the dynamics. The fundamental merit of Wilczek's discussion is to point out a fascinating problem in two-dimensional statistical mechanics (specified at low densities entirely by the magnitude of α mod 2π), whose solution would doubtless be most illuminating. A parallel exists in one dimension, where a Bose gas with repulsive two-body delta function interactions becomes a free Fermi gas for infinite repulsion.[12] In that case the exact solution is already known for intermediate repulsion, and can be interpreted as an interacting Fermi gas or as a Bose system with two types of Bose excitation.[13]

In sum, there are two ways to describe the spin and statistics of cyons. In the first, these systems are treated as the limiting case of solenoids becoming arbitrarily long (while remaining always parallel). Then continuity of the limit requires integer spin and Bose statistics, but the implications on phases of wave functions have no observable consequences because the cyon mass is infinite. If we abstract from the limit a two-dimensional system of finite-mass magnetic vortices combined with electric charges which feel the vortex vector potentials but do not interact with each other, then we are entitled to label the spin by the localized angular momentum $\overline{M}_z$, ignoring the contribution $+e\Phi/2\pi$ dispersed to infinity. At the same time we may minimize the cyon-cyon vector potential by a suitable gauge transformation. Then we find that the statistics are Bose or Fermi according as $\overline{M}_z$ is an integer or half integer, but are ill defined for intermediate values of $\overline{M}_z$. One would like to know the solution in that regime, and to compare it with the one-dimensional analogue.[13]

A remark of Wilczek,[1] that a cyon made of an electron bound to a magnetic vortex in a superconductor would be a boson, suggests how finite-mass cyons could conceivably arise. In a uniformly thin defect-free sheet of type-I superconductor, magnetic vortices might move freely with modest effective mass. If, furthermore, one electron could be trapped inside each vortex, and if this gave the vortex an effective charge equal to that of an electron (not obvious because of shielding effects), then the resulting system could behave like a Bose gas of cyons. If the sheet took the form of a disc with a hole in the middle, then Bose condensation of the cyon gas would inhibit magnetic flux through the hole equal to odd integral multiples of the usual flux quantum in superconductors. The reason is that the new gas would supply carriers with charge e rather than $2e$. Though this description is highly speculative, it suggests that experiments with superconducting sheets could yield interesting phenomena.

Let us go on to the statistics of Witten dyons.[3] The question is whether the statistics could be a function of the chiral angle θ. The most straightforward approach is to recall that statistics are equivalent to the statement of allowed and forbidden angular momenta for a system of two indistinguishable dyons. If we assume, for simplicity, that the monopoles have no internal angular momentum, the same must hold for the dyons, since angular momentum eigenvalues are discrete and cannot change continuously with chiral angle.

Therefore, the orbital angular momentum in the center-of-mass frame is the generator of rotations on the two-dyon system. This surely commutes with chiral transformations, which do not pick out a direction in space. Hence, the allowed angular momentum values, or equally well the statistics, may not change with θ.

A completely different approach with the same result is to emulate the procedure used in discussing statistics of dyons composed of isolable charges e and poles g.[9] In this case, there appears inevitably a vector potential for dyon-dyon interactions which, in a particular convention, may be written $\vec{A} = 2g\nabla\varphi$, where φ is the azimuthal angle of the dyon-dyon relative displacement with respect to some fixed direction. Provided that $2eg$ is an integer, this $\vec{A}$ may be eliminated by the gauge transformation phase factor $\psi = e^{2ieg\varphi}\psi'$, where ψ' feels no vector potential. Interchanging the dyons gives for ψ' a sign $(-1)^{2eg}$ times the sign for ψ. For odd-half-integer values of eg, this sign is just what is needed to obtain the usual connection of spin and statistics.

Since Witten dyons are not assembled from free poles and charges, the vector potential mentioned above might not be present for them. Indeed, it cannot be, since for nonintegral values of $2qg$ the axis about which φ varies would be observable by the Aharonov-Bohm effect in dyon-dyon scattering. Therefore, the phase factor which automatically appears to maintain the conventional spin-statistics connection for composite dyons is just as firmly excluded for Witten dyons, where it could only have spoiled the connection.

Wilczek[2] arrives at the same conclusion by assuming that the vector potential $\vec{A}$ *is* present, but (here we are making an interpretation of sketchy remarks in his paper) requiring that the wave function ψ' be single valued. Therefore, going from $\psi \rightarrow \psi'$ to eliminate the vector potential takes something normally forbidden, a multivalued gauge transformation. Since statistics are meaningless for the multivalued wave function ψ, they must be applied to ψ' directly, which is another way of saying that $\vec{A}$ should not appear in the first place.

The first discussion of dyons not involving isolable charges was that of Jackiw and Rebbi,[14] who noted that monopoles could possess vacuum polarization of plus or minus half an electron (*ipso facto* a half unit of electron charge). The remarks above about the dyon-dyon vector potential which is neither required nor allowed apply equally well to this case. Since these dyons have the same spin as if the fermions were not coupled, it is comforting that the dyon statistics are unchanged. The lesson seems to be that long-range phase effects appear only when needed, so that dyon statistics are always normal if monopole statistics are.

The topics discussed here are but a portion of those which Wilczek addresses. Among the most interesting of those remaining are instructive comments on dyon stability against charged particle emission, and the startling but compelling proposal that dyons could catalyze baryon decay at a high rate.[15]

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