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Bell Inequalities with a Range of Violation that Does Not Diminish as the Spin Becomes Arbitrarily Large

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Some new, powerful, and very simple Bell inequalities are derived. They establish that for two spin-s particles in the singlet state, the quantum-theoretic Einstein-Podolsky-Rosen correlations for spin measurements along all pairs of directions from a set of N axes are incompatible with local realism for any set of distinct coplanar axes when $N=3$, and for any set of distinct axes whatever when $N=4$, with the possible exception of sets restricted by one of the constraints $\hat{a}\pm\hat{b}\pm\hat{c}\pm\hat{d}=0$. These results hold for any value of the spin.

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In the spin-s generalization of Bohm's[1] spin-$\frac{1}{2}$ version of the Einstein-Podolsky-Rosen[2] experiment, the spin components m and m' of two spin-s particles in a spin singlet state φ are measured along directions $\hat{a}$ and $\hat{a}'$.[3-5] The quantum-theoretic joint distribution $q_{\hat{a}\hat{a}'}(m,m')$ vanishes whenever $\hat{a}=\hat{a}'$ unless $m'=-m$. It is thus possible to predict with certainty the result of measuring the spin of one particle along any direction, by measuring the spin of the other (which can be far away) along that direction. Einstein, Podolsky, and Rosen would account for this by an assumption of "local realism"—namely that the particles are characterized by functions μ and μ' that specify the outcome of spin measurements along any direction, and are related by

$$\mu'(\hat{a})=-\mu(\hat{a}). \quad (1)$$

Local realism is, of course, grossly incompatible with the metaphysical position of the quantum theory. In 1964 Bell[6] showed for $s=\frac{1}{2}$ that local realism is also numerically incompatible with the quantum theory, in that it requires groups of three joint distributions for three sets of correlation experiments, $q_{\hat{a}\hat{b}}$, $q_{\hat{a}\hat{c}}$, and $q_{\hat{b}\hat{c}}$, to satisfy an inequality that the quantum-theoretic distributions violate for appropriate sets of three distinct[7] axes $\hat{a}$, $\hat{b}$, and $\hat{c}$.

It was recently shown[3] that local realism implies similar inequalities for these three distributions in the general spin-s case, and that for any value of s, there are sets of coplanar directions $\hat{a}$, $\hat{b}$, $\hat{c}$, for which the spin-s inequalities are violated by the quantum-theoretic distributions. It was noted that the range of coplanar axes $\hat{a}$, $\hat{b}$, $\hat{c}$ for which the inequalities are violated shrinks to zero in the classical ($s\to\infty$) limit. Subsequently,[4] however, theoretical evidence was found that this vanishing of the range of violation with increasing s might be an artifact of the particular *ad hoc* numerical inequality used in Ref. 3 to generalize Bell's argument.

Quite recently Ögren[5] has derived several new sets of spin-s Bell inequalities for coplanar directions $\hat{a}$, $\hat{b}$, and $\hat{c}$. He finds a wider range of coplanar axes for which the quantum-theoretic distributions $q_{\hat{a}\hat{b}}$, $q_{\hat{b}\hat{c}}$, and $q_{\hat{a}\hat{c}}$ violate his inequalities, though this range also approaches zero in the limit of infinite s, the appropriate angular measure, in one of his cases, vanishing as $s^{-1/2}$ in contrast to an s^{-1} behavior for the inequalities of Ref. 3.

We confirm below that any such vanishing of the range of violating geometries as the classical limit is approached must indeed be an artifact of the particular analytical trick employed in the argument. We show that local realism implies another quite simple set of Bell inequalities which are violated by the quantum-theoretic distributions $q_{\hat a\hat b}$, $q_{\hat b\hat c}$, and $q_{\hat c\hat a}$ for *any* set of three distinct coplanar axes, $\hat a$, $\hat b$, $\hat c$, *whatever the value of the spin.*

Thus for three distinct coplanar axes the extraordinary aspect of the quantum-theoretic pair distributions q associated with them holds for any choice of axes and for any value of the spin. It is known,[8] however, that the general three-axis configuration space has regions of nonvanishing volume which contain sets of noncoplanar axes for which no three-axis Bell inequality of any kind can be violated by the quantum-theoretic distributions. We shall show below that if one admits a fourth axis, the power of the inequalities is not subject to such a limitation: We show that local realism implies a family of four-axis Bell inequalities satisfied by the six distributions $q_{\hat a\hat b}$, $q_{\hat a\hat c}$, $q_{\hat a\hat d}$, $q_{\hat b\hat c}$, $q_{\hat b\hat d}$, and $q_{\hat c\hat d}$, which hold for *any* spin s, but are violated by the quantum-theoretic distributions for all four-axis sets $\hat a$, $\hat b$, $\hat c$, $\hat d$ except for a simple zero-volume subset of the four-axis configuration space.

Our results (and those of Refs. 3–6) are based on the fact that if the particles were indeed characterized by functions of direction μ and μ' specifying the subsequent results of measuring their spins along any axes, then each separate particle could be characterized by an N-axis distribution $p_{\hat a_1\cdots\hat a_N}(m_1,\ldots,m_n)$ giving the probability that the function characterizing that particle had the values m_i in the directions $\hat a_i$, $i=1,\ldots,n$. If such functions existed, it would follow from the relation (1) between μ and μ' that for $N=2$ they would be given (for particle 1) in terms of the quantum-theoretic distribution q characterizing the pair correlation experiment by

$$p_{\hat a_1\hat a_2}(m_1,m_2)=q_{\hat a_1\hat a_2}(m_1,-m_2). \qquad (2)$$

The quantum theory, of course, emphatically denies that the distributions p_N have any meaning for a single particle, since they purport to describe the joint distribution of several distinct components of a single spin operator. The point of Bell's argument and the argument that follows is that the existence of such distributions is not only philosophically incompatible with the quantum theory, but also quantitatively incompatible with its numerical predictions. The strategy of such arguments is as follows.

From a purely statistical point of view there is nothing objectionable about the two-axis distributions $p_{\hat a_1\hat a_2}$ defined by (2); they are nonnegative, return the observed one-axis distributions as marginals, and, in conjunction with the assumption (1) of local realism, account for the observed two-particle correlations contained in $q_{\hat a_1\hat a_2}$. The difficulty, on purely statistical grounds, comes from the additional requirement that there should be N-axis distributions p with $N>2$ that return the two-axis distributions as marginals. Thus for any three axes $\hat a$, $\hat b$, $\hat c$, there should exist a nonnegative function $p_{\hat a\hat b\hat c}$ satisfying

$$p_{\hat a\hat b}(m_1,m_2)=\textstyle\sum_{m_3}p_{\hat a\hat b\hat c}(m_1,m_2,m_3) \qquad (3)$$

and the analogous two equations for the distributions $p_{\hat a\hat c}$ and $p_{\hat b\hat c}$. By a three-axis Bell inequality we mean a condition on three axes $\hat a$, $\hat b$, $\hat c$, which is necessary (but not, in general, sufficient[9]) for the existence of any nonnegative $p_{\hat a\hat b\hat c}$ satisfying the relations (3). The condition applies to the observed distributions $q_{\hat a\hat b}$, $q_{\hat b\hat c}$, and $q_{\hat a\hat c}$, since these are directly related to the hypothetical $p_{\hat a\hat b\hat c}$ through (2) and (3). The condition comes in the form of an inequality since it is easy to construct functions $p_{\hat a\hat b\hat c}$ satisfying the conditions (3); the difficulty lies in finding nonnegative functions.

The Bell inequalities given below establish for any spin s that for any distinct coplanar axes $\hat a$, $\hat b$, $\hat c$, the quantum-theoretic distributions $q_{\hat a\hat b}$, $q_{\hat a\hat c}$, and $q_{\hat b\hat c}$ are numerically incompatible with the existence of any nonnegative function $p_{\hat a\hat b\hat c}$. They also show for any spin s that a necessary condition for there to be a four-axis distribution $p_{\hat a\hat b\hat c\hat d}$ satisfying

$$p_{\hat a\hat b}(m_1,m_2)=\sum_{m_3,m_4}p_{\hat a\hat b\hat c\hat d}(m_1,m_2,m_3,m_4), \qquad (4)$$

and the analogous five equations for $p_{\hat a\hat c}$, $p_{\hat a\hat d}$, $p_{\hat b\hat c}$, $p_{\hat b\hat d}$, and $p_{\hat c\hat d}$, is that the four axes satisfy one of the eight constraints $\hat a\pm\hat b\pm\hat c\pm\hat d=0$.

The only properties of the quantum-theoretic distributions needed to establish these results are the following:

(a) *Bilinearity of the correlation functions.* —The mean of the product mm',

$$\langle mm'\rangle_{\hat a\hat b}=\textstyle\sum_{mm'}mm'\,q_{\hat a\hat b}(m,m') =(\varphi,(\vec S^1\cdot\hat a)(\vec S^2\cdot\hat b)\varphi), \qquad (5)$$

is explicitly bilinear in the axes $\hat{a}$ and $\hat{b}$.

(b) *Rotational invariance of the singlet state.* —This requires that the bilinear form in (5) must in fact be proportional to $\hat{a}\cdot\hat{b}$ [10]:

$$\langle mm'\rangle_{\hat{a}\hat{b}} = K_s(\hat{a}\cdot\hat{b}). \qquad (6)$$

(c) *One-particle distributions independent of the far detector.*—This requires that

$$\textstyle\sum_{m'} q_{\hat{a}\hat{b}}(m,m') \text{ is independent of } \hat{b}. \qquad (7)$$

(d) *Perfect correlations for parallel axes.* —This is the basis for the Einstein-Podolsky-Rosen argument:

$$q_{\hat{a},\hat{a}}(m,m') = \delta_{m,-m'}. \qquad (8)$$

(e) *Positivity of the extremal distributions.* —We have

$$q_{\hat{a},\hat{b}}(m,m') \neq 0 \text{ if } |m| \text{ or } |m'| = s, \quad \hat{a} \neq \pm\hat{b}. \qquad (9)$$

This follows from the explicit form for q, which is just the binomial distribution when one of the variables is fixed at $\pm s$.

Note that of the five properties (a)–(e), only (e) fails to hold for the classical joint distribution.

The proofs of these results are surprisingly simple:

(1) *Nonexistence of distributions for three distinct coplanar axes.*—If $\hat{a}$, $\hat{b}$, and $\hat{c}$ are distinct coplanar axes then there is a relation

$$A\hat{a} + B\hat{b} + C\hat{c} = 0 \qquad (10)$$

with nonzero coefficients. Label the axes so that $|A| \geq |B| \geq |C|$. Given any candidate for a three-axis distribution, $p_{\hat{a}\hat{b}\hat{c}}$, define

$$f_{\hat{a}\hat{b}\hat{c}} = \sum_{m_1 m_2 m_3} (Am_1 + Bm_2 + Cm_3)^2 p_{\hat{a}\hat{b}\hat{c}}(m_1,m_2,m_3). \qquad (11)$$

Since the sum is term by term nonnegative it is bounded below by any partial sum. In particular,

$$f_{\hat{a}\hat{b}\hat{c}} \geq \textstyle\sum_{m_3}(|A|s + |B|s + Cm_3)^2 p_{\hat{a}\hat{b}\hat{c}}(\epsilon_A s, \epsilon_B s, m_3), \qquad (12)$$

where $\epsilon_X = X/|X|$. If we replace the squared trinomial by its lower bound $(|A| + |B| - |C|)^2 s^2$, then the sum on m_3 gives a two-axis distribution, and we arrive at the lower bound:

$$f_{\hat{a}\hat{b}\hat{c}} \geq (|A| + |B| - |C|)^2 s^2 q_{\hat{a}\hat{b}}(\epsilon_A s, -\epsilon_B s). \qquad (13)$$

Property (e) requires this lower bound to exceed zero. Explicitly squaring the trinomial in Eq. (11), however, yields nine terms, each of which can be expressed in terms of two-axis distributions, and hence in terms of q. Properties (a)–(d) permit each term to be evaluated, and the nine can then be recombined to give[11]

$$f_{\hat{a}\hat{b}\hat{c}} = -K_s(A\hat{a} + B\hat{b} + C\hat{c})^2, \qquad (14)$$

which vanishes in view of (10). There can therefore be no $p_{\hat{a}\hat{b}\hat{c}}$.

(2) *Nonexistence of distributions for any four distinct axes satisfying* $\hat{a}\pm\hat{b}\pm\hat{c}\pm\hat{d}\neq 0$.—If $p_{\hat{a}\hat{b}\hat{c}\hat{d}}$ exists then so do the three-axis distributions obtained from it as marginals, and therefore by the above result no three of the axes can lie in a single plane. There therefore exists a relation

$$A\hat{a} + B\hat{b} + C\hat{c} + D\hat{d} = 0 \qquad (15)$$

with nonzero coefficients. Label the axes so that $|A| \geq |B| \geq |C| \geq |D|$. Define

$$f_{\hat{a}\hat{b}\hat{c}\hat{d}} = \sum_{m_1 \cdots m_4} (Am_1 + Bm_2 + Cm_3 + Dm_4)^2 p_{\hat{a}\hat{b}\hat{c}\hat{d}}(m_1,m_2,m_3,m_4). \qquad (16)$$

As in the three-axis case, we now argue that

$$f_{abcd} \geq \sum_{m_3 m_4}(|A|s + |B|s + Cm_3 + Dm_4)^2 p_{\hat{a}\hat{b}\hat{c}\hat{d}}(\epsilon_A s, \epsilon_B s, m_3 m_4) \geq (|A| + |B| - |C| - |D|)^2 s^2 q_{\hat{a}\hat{b}}(\epsilon_A s, \epsilon_B s). \qquad (17)$$

In view of property (e) this last lower bound can only be zero if $|A| = |B| = |C| = |D|$. But the procedure that led to (14) now gives

$$f_{\hat{a}\hat{b}\hat{c}\hat{d}} = K_s(A\hat{a} + B\hat{b} + C\hat{c} + D\hat{d})^2, \qquad (18)$$

which vanishes in view of (15). Therefore $p_{\hat{a}\hat{b}\hat{c}\hat{d}}$ can exist only when the relation (15) reduces to one of the forms $\hat{a}\pm\hat{b}\pm\hat{c}\pm\hat{d} = 0$.

Note that for spin $\frac{1}{2}$ the excluded geometries are not artifacts. The four-axis distribution

$$p_{\hat{a}\hat{b}\hat{c}\hat{d}}(m_1,m_2,m_3,m_4) = -\tfrac{1}{16} + m_1 m_2 m_3 m_4 + \tfrac{1}{8}(m_1\hat{a} + m_2\hat{b} + m_3\hat{c} + m_4\hat{d})^2 \qquad (19)$$

gives all the correct two-axis distributions and is nonnegative when $\hat{a}+\hat{b}+\hat{c}+\hat{d}=0$.[12]

For $s>\frac{1}{2}$, four-axis distributions do not in general exist even in the excluded geometries. For example, let $s=1$ and let $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ point from the center to the vertices of a tetrahedron, so that (15) holds with $A=B=C=D=1$. In this geometry the two-axis distributions are independent of the axis pair, and are given by $p(m_1, m_2)=\frac{1}{27}$, $m_1=m_2$; $p(m_1,m_2)=\frac{4}{27}$, $m_1\neq m_2$. It suffices to consider four-axis distributions symmetric in $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$,[13] and hence independent of the order of the four arguments m_i. The vanishing of $f_{\hat{a}\hat{b}\hat{c}\hat{d}}$ required by (18) means, in view of the definition (16), that the four-axis p must vanish unless $m_1+m_2+m_3+m_4=0$. We therefore have $\frac{1}{27}=p(1,1)=p(1,1,-1,-1)$. Furthermore, $\frac{4}{27}=p(1,-1)=p(1,-1,0,0)+2p(1,-1,1,-1)$, and so we must have $p(1,-1,0,0)=\frac{2}{27}$. But $p(0,0)=p(0,0,0,0)+2p(0,0,1,-1)$, which requires $p(0,0)$ to exceed $\frac{4}{27}$. Since the actual value is $\frac{1}{27}$, there can be no four-axis distribution.[14]

We conclude that the necessary emergence of local realism in the classical limit is not signaled by the vanishing of the range of geometries that violate Bell inequalities. For the inequalities derived here it follows, instead, from the vanishing magnitude of those violations.[15] In this regard it should be noted that there are lower bounds that vanish far more slowly with increasing s than the one given in (13). We consider this point and the generalization of our approach to the weaker form of local realism tested by the inequalities of Clauser and Horne[16] in a subsequent publication.[17]

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[7]We say $\hat{n}$ and $\hat{n}'$ are "distinct" if $\hat{n}\neq\pm\hat{n}'$.

[8]For spin-$\frac{1}{2}$ this was shown by E. P. Wigner, Am. J. Phys. 38, 1005 (1970). For the spin-1 case see Ref. 4, Sec. 5.2.

[9]Wigner (Ref. 8) gives necessary *and* sufficient conditions for spin $\frac{1}{2}$; Mermin and Schwarz (Ref. 4, Sec. 5.2), for spin 1.

[10]It is easy to show that $K_s=-s(s+1)/3$, but this is not exploited in any of our arguments.

[11]To evaluate the three terms in which only a single m appears using only properties (a)–(d), use (7) to set the two axes equal, (8) to replace m^2 by $-mm'$, and (6) to give the result.

[12]For the other exceptional geometries simply reverse the sign of each m_i in (19) corresponding to each axis with a negative sign in $\hat{a}\pm\hat{b}\pm\hat{c}\pm\hat{d}=0$. These four-axis distributions can be shown to be the only ones possible.

[13]If an asymmetric one existed, symmetrizing it in the four axes would yield a symmetric one with all the required properties.

[14]It is shown in Ref. 4 [see Eq. (6.8)] that the three-axis spin-1 distributions do exist when the set $\hat{a}$, $\hat{b}$, $\hat{c}$ contains three of the four tetrahedral directions. For spin 1, four tetrahedral axes therefore provide an example of a geometry for which no four-axis distribution exists, even though all possible Bell inequalities must be satisfied for any three-axis subset of the four.

[15]The vanishing of the magnitude, however, may also be an artifact of the particular inequalities we have derived here, since they all follow from taking the mean of slowly varying functions of the spin components m, and are therefore insensitive to the rapidly varying parts of the quantum-theoretic distributions $q_{\hat{a}\hat{a}'}(m,m')$.

[16]J. F. Clauser and M. A. Horne, Phys. Rev. D 10, 526 (1974).

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