PHYSICAL REVIEW LETTERS

VOLUME 49 | 13 SEPTEMBER 1982 | NUMBER 11

Percolation Theory of Nonlinear Circuit Elements

Stephen W. Kenkel[(a)] and Joseph P. Straley[(b)]

Department of Physics and Astronomy, University of Kentucky, Lexington, Kentucky 40506

(Received 8 March 1982)

Percolation theory is extended to the case of circuit elements with nonlinear I-V characteristics, particularly the special cases $V = I^{\alpha} r$, which form the universality classes. Near the percolation threshold the "conductance" $I/V^{1/\alpha}$ vanishes like $(p - p_c)^t$, where t depends on α and dimensionality. The Skal–Shklovskii–De Gennes model gives $t(\alpha) = (d-1)\nu + (\varphi - \nu)/\alpha$, where φ is only weakly dependent on α and d, and approaches unity at $d = 6$. Renormalization methods are used to study the α dependence of t in two dimensions.

PACS numbers: 05.40.+j, 84.20.Pc

A network built of randomly chosen conducting and nonconducting elements has a conductivity which vanishes as the percolation threshold is approached.[1] The behavior near threshold is described by a universal power law; the dimensionality dependence of the exponent has been given extensive study.[2-4]

The problem can be generalized: Determine the relationship between average current density and average field in a network of randomly chosen *nonlinear* elements. The generalization brings with it two complications that prevent direct transcription of previous theory: (1) The superposition principle no longer applies. This rules out a number of short cuts; in particular, effective medium theory is no longer the simplest approach to the problem. (2) The I-V characteristic of a combination of elements may be different in functional form from that of the components. For example, consider a combination of identical circuit elements having two in series and a third connected parallel to these: If the individual elements have a singular point in their I-V characteristic at voltage V_0, the combination will have two singular points at V_0 and $2V_0$. Thus part of the problem is to determine the unknown functional form of the I-V characteristic of the network.

In view of the argument just given we can anticipate that in the limit of an infinite system the functional form will be smooth (except possibly at $V = 0$): If we build a large system by joining a few components into a module, joining modules into larger arrays, and so on, the number of singularities proliferates while their individual significance decreases.

The I-V characteristic of a network that is only slightly inhomogeneous will strongly resemble the average I-V characteristic of its component elements, but with the corners rounded off. Increasingly inhomogeneous networks will also have increasingly smooth I-V characteristics. As the percolation threshold is approached, we might anticipate the emergence of a class of "universal" I-V characteristics. Since the network at the percolation threshold is inhomogeneous at all length scales, we can see that a necessary feature of the asymptotic form is that a random network built of elements with this I-V characteristic must itself have the universal I-V curve.

The observation which we now put forward is that circuit elements which have a power-law relationship

$$V = r|I|^{\alpha}\,\mathrm{sgn}I \qquad (1)$$

do preserve the functional form when combined. Consider an arbitrary two-terminal network built of such elements, all obeying the same power law (the coefficients r may be chosen arbitrarily, however) and subjected to an arbitrary applied voltage V_{ext}. The resulting voltage and current distribution is determined by Kirchhoff's rules with Ohm's law replaced by Eq. (1). Now if all voltage differences, including the external voltage, are increased by a factor x and all currents increased by a factor $x^{1/\alpha}$, a new solution to Kirchhoff's rules is generated. The current from the voltage source is thus proportional to ${V_{\text{ext}}}^{1/\alpha}$, allowing us to generalize the concepts of conductance and resistance,

$$g = IV^{-1/\alpha}, \quad r = VI^{-\alpha}. \qquad (2)$$

Similarly, a uniform field in a homogeneous nonlinear medium of this type will give rise to a uniform current density, related by a conductivity $JE^{-1/\alpha}$.

In view of the discussion above, this result shows that the power-law conductors for any given α form a universality class of the nonlinear conductivity problem.

This argument cannot be extended to three-terminal networks, because the equivalence between Y-shaped and triangular networks that obtains for linear networks fails to generalize.[5]

Two special cases can be included as limiting cases of the class: bipolar Zener diodes (which do not conduct below V_0) are the case $\alpha \to 0$, and according to Eq. (2) the "resistance" of the device is V_0; saturating conductors (which will not carry a current higher than I_s) are approximately the case $\alpha \to \infty$, and the "conductance" of the device is I_s. The properties of a random network of saturable conductors has been discussed previously.[6-9]

The behavior of circuit elements with more general[10] I-V characteristics can be discussed in terms of the power-law conductors. For example, one might encounter a circuit element described by one power law for small applied voltage, and a different power law for large voltage:

$$I = \begin{cases} c(V/V_0)^{1/\alpha_1}\,\mathrm{sgn}V & \text{for } |V| < V_0, \\ c(V/V_0)^{1/\alpha_2}\,\mathrm{sgn}V & \text{for } |V| > V_0. \end{cases} \qquad (3)$$

It is useful to visualize the function plotted as $\ln I$ vs $\ln V$: two straight lines of differing slopes. For sufficiently small applied fields, the behavior of a randomly dilute network of these elements will be determined by α_1,

$$I \approx c(p-p_c)^{t_1}(V/V_0)^{1/\alpha_1}, \qquad (4)$$

whereas the large-field behavior is determined by α_2,

$$I \approx c(p-p_c)^{t_2}(V/V_0)^{1/\alpha_2}. \qquad (5)$$

On the logarithmic graph the two straight lines have been shifted down by differing amounts. The crossover voltage can be estimated by equating the two expressions

$$V_c = V_0(p-p_c)^{\alpha_1\alpha_2(t_1-t_2)/(\alpha_1-\alpha_2)}. \qquad (6)$$

This analysis, generalized to I-V characteristics which are arbitrary piecewise combinations of power laws, suggests that as p approaches p_c any given range of V becomes dominated by the power law with the largest $t(\alpha)$ among those originally represented. In this sense the power laws and their exponents are all we need to know.

This same example, however, shows that the asymptotic high- and low-field behavior does not change, which raises the question: What happens in between? In the simplest case (assumed above), there is a single crossover. In the limit $p \to p_c$, where only the cooperative aspects of the percolation problem are relevant, the behavior of the macroscopic system near V_c is given by a crossover function; analogy with other cooperative phenomena involving crossover suggests that this is a universal function depending only on dimensionality and the power laws that describe the asymptotes: The microscopic crossover behavior is irrelevant and renormalizes away. Multiple crossovers can also occur, but are independent of each other and thus lead to nothing new.

The study of the conductivity of a random network of power-law conductors near the percolation threshold has been carried out by generalization of methods previously employed for linear networks. The principal results are as follows (details will be published elsewhere):

(A) Skal and Shklovskii[11] and De Gennes[12] have given a useful model for a network near the percolation threshold: chains of resistance $\mathcal{L} \sim (p-p_c)^{-\varphi}$ which join at nodes separated by a distance $\xi \sim (p-p_c)^{-\nu}$. In an applied field E, the voltage drop across a chain is of order $E\xi$, giving a current $(E\xi/\mathcal{L})^{1/\alpha}$, and current density $J = (E\xi/$

$\mathcal{L})^{1/\alpha}\xi^{1-d}$. Then near threshold the conductivity $JE^{-1/\alpha}$ behaves like $(p-p_c)^t$ with

$$t=(d-1)\nu+(\varphi-\nu)/\alpha. \tag{7}$$

The exponent ν is determined by the geometry of the network and is well studied.[13,14] The exponent φ is approximately unity for all dimensionality and α. The special case $\alpha\to\infty$ of Eq. (7) has been given by Deutscher and Rappaport.[8]

(B) The result (7) is only expected to hold up to the upper critical dimensionality ($d^*=6$), where the exponents take on their high-dimensionality values. Above d^* the chains are well modeled as simple random walks, which implies $\nu^*=\frac{1}{2}$ and $\varphi^*=1$, giving $t^*=\frac{5}{2}+(2\alpha)^{-1}$. We have succeeded in verifying that this is the correct exponent for the Cayley-tree model, which is an endlessly branching network with no closed loops and is usually taken to be an infinite-dimensional system.

(C) The α dependence of t was studied with a renormalization method[15-17] in two dimensions. The results are shown in Fig. 1. The renormalization cell is shown in the inset; it is the "Wheatstone bridge" cell of Bernasconi,[16] which has the advantage that it is self-dual. It gives $p_c=\frac{1}{2}$ and $\nu=1.4277$. For any given configuration of bond conductivities we may calculate a conductance; for example, if all bond conductivities are unity, the cell conductance is $2^{1-1/\alpha}$. The cell conductivity is then defined to be the conductance divided by $2^{1-1/\alpha}$, so that a lattice of all unit conductors has unit conductivity.

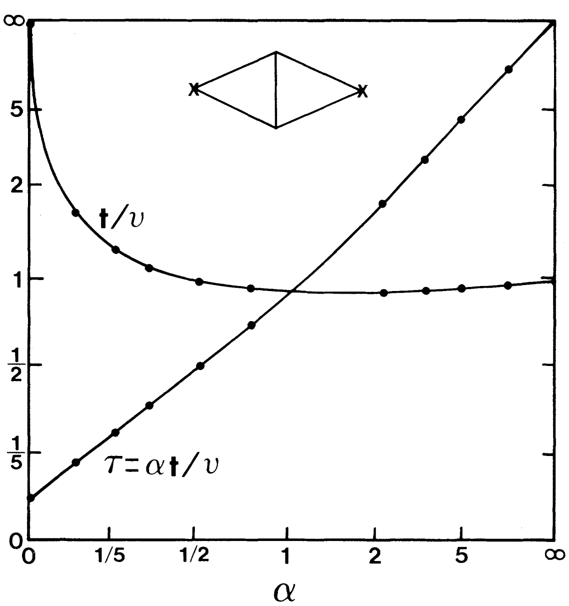

FIG. 1. α dependence of t and τ in two dimensions, according to a renormalization study. A nonlinear scale is used to allow the whole range of these quantities to be seen; it was generated by a linear plot of $t/(\nu+t)$ and $\tau/(\nu+\tau)$ against $\alpha/(1+\alpha)$. Inset: the renormalization cell used.

The conductivity exponents were calculated as follows. Several thousand configurations were constructed by choosing the five bonds of the cell from a conductivity distribution P, and the set of conductivities obtained formed a new distribution P'. This process, which constitutes the renormalization transformation, was repeated ten times. The average conductivity for the successive distributions decreases by a factor which provides an estimate for t:

$$2^{-t/\nu}=\langle\sigma\rangle_{P'}/\langle\sigma\rangle_P. \tag{8}$$

The saturating conductor and the Zener diode were also studied by this method; these results are entered in Fig. 1 as the $\alpha\to\infty$ limit of $t(\alpha)$ and the $\alpha\to 0$ limit of $\tau(\alpha)\equiv\alpha t(\alpha)$, respectively. The results are consistent with Eq. (7) with φ only weakly dependent on α.

Percolation effects in a few specific types of nonlinear conducting elements have been considered before,[6-9] but this is the first attempt to treat the general problem. Potential applications —beyond the obvious examples of randomly doped semiconductors and the ZnO varistor[18]—might be the flow of nonwetting fluids through finely porus material, the magnetic properties of ferromagnetic ceramics (with H and B replacing V and I), and random Josephson networks (where the supercurrent has nonlinear dependence on phase difference).

This work was supported by the National Science Foundation through Grants No. DMR 78-27023 and No. DMR 81-03099.

[(a)]Present address: Department of Physics and Astronomy, University of North Carolina, Chapel Hill, N. C. 27514.

[(b)]Present address: Department of Physics and Astronomy, University of Alabama, University, Ala. 35486.

[1]S. Kirkpatrick, Rev. Mod. Phys. 45, 574–588 (1973).

[2]J. P. Straley, in *Electrical Transport and Optical Properties of Inhomogeneous Media—1977*, edited by J. C. Garland and D. B. Tanner, AIP Conference Proceedings No. 40 (American Institute of Physics, New York, 1978), pp. 118–126.

[3]J. P. Straley, J. Phys. C 13, 819–822 (1980).
[4]J. P. Straley, J. Phys. C 15, 2333–2341 (1982).
[5]There are nonlinear systems that exhibit a "wye-delta" transformation [D. Frank and C. Lobb, Bull. Am. Phys. Soc. 27, 388 (1982), Abstract KN12], but these are also special isolated cases.
[6]D. A. Huse and R. A. Guyer, Phys. Rev. Lett. 43, 1163 (1979).
[7]S. Kirkpatrick, in *Inhomogeneous Superconductors —1979*, edited by D. U. Gubser *et al.*, AIP Conference Proceedings No. 58 (American Institute of Physics, New York, 1979), pp. 79–86.
[8]G. Deutscher and M. L. Rappaport, J. Phys. (Paris), Lett. 40, L219 (1979).
[9]C. J. Lobb and D. J. Frank, in Ref. 7, pp. 308–313.
[10]We suggest that the only limitation on the generalization is that the macroscopic conductivity be well defined. However, there are gray areas, such as multiple-valued I-V characteristics (which give rise to hysteretic or oscillatory behavior) and functional forms which are not antisymmetric in V (which may involve crossover with the directed percolation problem).
[11]A. S. Skal and B. I. Shklovskii, Fiz. Tekh. Poluprov. 8, 1586–1592 (1974) [Sov. Phys. Semicond. 8, 1029–1032 (1975)].
[12]P. G. De Gennes, J. Phys. (Paris), Lett. 37, L1–2 (1976).
[13]S. Kirkpatrick, Phys. Rev. Lett. 27, 1722–1725 (1971).
[14]A. G. Dunn, J. W. Essam, and D. Ritchie, J. Phys. C 8, 4219–4235 (1975).
[15]R. B. Stinchcombe and B.P. Watson, J. Phys. C 9, 3221 (1976).
[16]J. Bernasconi, Phys. Rev. B 18, 2185–2191 (1978).
[17]J. P. Straley, J. Phys. C 10, 1903–1909 (1977), and 12, 3711–3716 (1979).
[18]M. Matsuoka, T. Masuyama, and Y. Iida, J. Jpn. Soc. Appl. Phys., Suppl. 39, 94 (1970).