Roughening and Lower Critical Dimension in the Random-Field Ising Model

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It is argued on the basis of a new interface model that the lower critical dimension of random-field Ising systems is two, in agreement with simple domain estimates.

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The lower critical dimension, d_c, of the Ising model in a random magnetic field (RFIM) has been a puzzle for some years. Simple, physical domain-wall arguments[1] at zero temperature (T) suggest that random fields destroy the ferromagnetic order whenever $d<2$; that is, $d_c=2$. On the other hand, arguments based on the equivalence, order by order in perturbation theory, between Ginzburg-Landau models with random fields in d dimensions and those without random fields in $d-2$ dimensions suggest[2] that d_c is three for the RFIM, two more than the d_c for the pure Ising problem.

Very recently calculations based on interface models of the RFIM,[3,4] analogous to those used to study the pure Ising model[5] for $d=1+\epsilon$, have supported the conclusion $d_c=3$. A crucial element of this ingenious line of reasoning is the assertion[3] that at $d=3$ the interface between domains of up and down spins in the RFIM is rough.[6] It is argued[7] that the failure to account for this roughness in the original domain estimates[1] invalidates those estimates. Moreover, experiments on various physical realizations of the RFIM[8] seem consistent with $d_c=3$.

In this paper we reconsider the interface approach to the RFIM, relying upon neither the replica method[3] nor the supersymmetry arguments[4] employed in the earlier treatments. Our starting continuum interface Hamiltonian differs significantly from those of the earlier calculations in that it is a nonanalytic function of the interface coordinates, $f(\vec{x})$. Simple power counting shows that $d_c=2$ for this Hamiltonian, in agreement with the original[1] domain estimate. We argue that this conclusion is consistent with the roughness of the interface at $d=3$; indeed, we find that the width, w, of the RFIM interface varies as L^x in d dimensions, where $x\equiv(5-d)/3$ and L is the linear dimension of the system. The interface is therefore rough (i.e., $x>0$) whenever $d<5$, in agreement with Pytte *et al.*[3,9] However, if $d>2$ then $x<1$ and $w/L\to 0$ as $L\to\infty$. For $d>2$ the interface width thus diverges more slowly than L: The interface is effectively smooth. Note that Pytte *et al.*[3] find $x=(5-d)/2$, whereupon $w\geq L$ for $d\leq 3$, consistent with their conclusion $d_c=3$.

Our starting point is the continuum Hamiltonian describing the $(d-1)$-dimensional interface between one domain of discrete Ising spins pointing up and one down. Let the shape of the interface be defined by $z=f(\vec{x})$ [$\vec{x}$ designating the $(d-1)$ interface coordinates] and let $h(\vec{x},z)$ denote the field at $(\vec{x},z)$. The Hamiltonian divided by the temperature is

$$H/T=T^{-1}\int d^{d-1}x\{J[1+(\nabla f)^2]^{1/2}+\int_0^{f(\vec{x})}h(\vec{x},z)dz\}. \qquad (1)$$

J is the exchange interaction strength, and $\int d^{d-1}x[1+(\nabla f)^2]^{1/2}$ the area of the interface. The field energy is arbitrarily chosen to be zero at $f=0$.

Note that since $h(\vec{x},z)$ is a random function of z, H is a nonanalytic function of $f(\vec{x})$. This nonanalyticity becomes more explicit in the replica formalism; choosing, e.g., at each point $(\vec{x},z)$ a Gaussian probability distribution of width $\sqrt{\Delta}$ for the random field one obtains, following a trivial integration, the replica Hamiltonian[10]

$$\frac{1}{T}H_n=\frac{1}{T}\int d^{d-1}x\left(J\sum_{\alpha=1}^{n}[1+(\nabla f_\alpha)^2]^{1/2}-\frac{\Delta}{T}\sum_{\alpha,\beta=1}^{n}\theta(f_\alpha f_\beta)\min(|f_\alpha|,|f_\beta|)\right), \qquad (2)$$

for $n\to 0$; here $\theta(x)=1,\frac{1}{2},0$ for $x>0$, $x=0$, $x<0$. The relationship between (2) and the analytic Hamiltonian of Ref. 3, which was derived directly in the replica language on the basis of symmetry considerations without reference to a nonreplicated interface Hamiltonian [such as (1)] in which the random field appears explicitly, is far from transparent. However, an intriguing connection between the two approaches follows from consideration of (1) in the case where the random fields $h(\vec{x},z)$ are independent of z; i.e., $h(\vec{x},z)=h(\vec{x})$. The second term of (1) for this "random rod" model is analytic: $f(\vec{x})h(\vec{x})$. The replica method applied to this model with $h(\vec{x})$ distributed according to a Gaussian yields the interface Hamiltonian

$$\frac{1}{T}H_n^{\mathrm{RR}}=\frac{1}{T}\int d^{d-1}x\left(J\sum_{\alpha=1}^{n}[1+(\nabla f_\alpha)^2]^{1/2}-\frac{\Delta}{T}\sum_{\alpha,\beta=1}^{n}f_\alpha(\vec{x})f_\beta(\vec{x})\right). \quad (3)$$

The final term of (3) is identical to the lowest-order nontrivial interaction term in the model of Ref. 3. Trivial dimensional considerations[3] show that both models have lower critical dimension $d_c=3$. Moreover, direct application of the domain argument[1] to the random-rod model likewise predicts $d_c=3$. To see this, imagine creating a domain of linear size L of down spins in a d-dimensional random-rod model assumed ferromagnetically ordered in the up direction. For large L the surface-energy cost of such a domain is, as usual, proportional to L^{d-1}, while the field energy goes like $LL^{(d-1)/2}\sim L^{(d+1)/2}$ (which is larger than the $L^{d/2}$ in an RFIM). It is thus energetically favorable to create large domains that destroy ferromagnetic order whenever $(d+1)/2>d-1$ or $d<3$; i.e., $d_c=3$ for the random-rod problem.

The connection described above between (3) and the Hamiltonian studied in Ref. 3 suggests that the latter may be more appropriate to the random-rod model than to the RFIM. This would explain the result $d_c=3$ of that work. However, since the complete Hamiltonian of Ref. 3 is considerably more complicated than (3) (and has different symmetry properties), it is unclear that the two models are equivalent.

That $d_c=2$ for the Hamiltonian (2) follows from an elementary scaling argument: Under the scale transformation $\vec{x}=b\vec{x}'$, the length $f(\vec{x})$ also transforms as $f(\vec{x})=bf'(\vec{x}')$, whereupon the definitions

$$T'=b^{1-d}T,\quad \Delta'=b^{2-d}\Delta \quad (4)$$

preserve the functional form of H_n. Equations (4) (the lowest-order renormalization-group recursion relations) show that Δ is an irrelevant variable for $d>2$; therefore $d_c=2$. At first glance this result seems inconsistent with perturbation theory performed with (2) [or equivalently, (1)]. If, e.g., one expands the surface tension $\sigma\equiv -T\ln\mathrm{Tr}\exp(-H_n/T)/nL^{d-1}$ (L being the linear dimension of the system) in powers of Δ in the $n=0$ limit, one obtains

$$\sigma=\sigma_0-AJ^{-1/2}L^{(3-d)/2}(\Delta/T^{3/2})[1+O(L^{(d+1)/2}\Delta J^{-1/2}T^{-3/2})], \quad (5)$$

where σ_0 is the surface tension of the pure system ($\Delta=0$) and A is a numerical constant. Since for all $d\le 3$ the term of order Δ diverges for $L\to\infty$, (5) suggests (or is at least consistent with) $d_c=3$ rather than $d_c=2$. Were $d_c=3$ in the RFIM one would, for all $d<3$, expect σ to drop discontinuously from σ_0 at $\Delta=0$ to zero for all $\Delta>0$; any attempt to expand σ in powers of Δ should therefore fail, as does (5), for $d<3$. The failure of perturbation theory for $d<3$ does not necessarily imply $d_c=3$, however. Indeed, we assert on the basis of a renormalization-group estimate that, at low T for $d>2$, $\sigma-\sigma_0\sim\Delta^y$ as $\Delta\to 0$ with $y=2(d-1)/(d+1)$; thus $\frac{2}{3}\le y\le 1$ for $2\le d\le 3$. Since $y<1$ for $d<$ an expansion such as (5) in integral powers of Δ *must* break down, with no implication that the RFIM is disordered for $\Delta>0$.

The calculation, of which only the barest outline can be presented here, is similar in spirit to the original phase-space cell analysis of Wilson.[7,11] Given a particular set of random fields, $h(\vec{x},z)$, we attempt to account approximately for fluctuations of the interface of progressively longer wavelength. We imagine $f(\vec{x})$ expanded in terms of an orthonormal set of functions $\varphi_\lambda(\vec{x})$: $f=\sum_\lambda\varphi_\lambda(\vec{x})q_\lambda$. The $\varphi_\lambda(\vec{x})$ are chosen to be "wave packets" with reasonably well-defined locations and magnitudes of $\nabla\varphi_\lambda(\vec{x})$ (i.e., "momenta"). Now substitute in (1) $f=f_0+f_1$, where f_1 is "slowly varying," i.e., essentially flat over length scales $\lesssim ba$, while f_0 varies over scales between a and ba. Here a is the short-distance cutoff of the theory and b (>1) the scale factor.[11] The task is now to calculate, for given fixed $f_1(\vec{x})$ (taken as constant over the region occupied by a given packet of size $\lesssim ba$), the contribution to the surface tension due to f_0, i.e., to variations in the interface over the shortest length scales.

This contribution can be estimated for each of the smallest wave packets independently; an elementary calculation (equivalent in essence to the domain argument[1] applied to a random system of size ba) gives the energy per unit area, H_{0b}, of a given packet as a function of its coefficient q (the subscripts λ are suppressed and we now adopt units in which $a=1$ to eliminate the superfluous dependence of our formulas on a):

$$b^{d-1}H_{0b} \sim Jb^{-2}(q^2 - CJ^{-1}\Delta^{1/2}|q|^{1/2}b^{(d+7)/4}). \quad (6)$$

Here C is a function of q, random in both magnitude and sign, of order unity. Since any wave packet φ localized in a region of linear size b is normalized so that $\int d^{d-1}x\,\varphi^2 = 1$, $\varphi \sim b^{-(d-1)/2}$ and $\nabla\varphi \sim b^{-(d+1)/2}$. The two terms of (6) are thus simply interpreted as an exchange or boundary term $[b^{d-1}(\nabla\varphi)^2q^2]$ and a random-field volume energy $(b^{d-1}|q\varphi|)^{1/2}$ for the packet φ. The corresponding thermally averaged surface tension σ_{0b} is then

$$-b^{-d+1}T\ln[\int dq\,\exp(-b^{d-1}H_{0b}/T)].$$

Having computed σ_{0b} one obtains from (4) the once-renormalized version of Hamiltonian (1) and repeats the calculation just described, thereby determining the surface tension due to fluctuations on scales between b and b^2. The total surface tension, σ, resulting from many such iterations is then expressed as the series

$$\sigma(T,\Delta) = \sum_{l=1}^{l_L} \sigma_{0b}(T_l,\Delta_l), \quad (7)$$

where $l_L \equiv \ln L/\ln b$, and $T_l = Tb^{l(1-d)}$ and $\Delta_l = \Delta b^{l(2-d)}$ are the l-times-iterated versions of T and Δ, respectively.

At $T=0$ the ground-state surface tension $E_{0b} \equiv \sigma_{0b}(T=0,\Delta)$ is determined by minimizing H_{0b} with respect to q. One finds, from (6) and (7), that the total ground-state surface tension $\sigma_0 \equiv \sigma(T=0,\Delta)$ is then

$$\sigma_0 = J - J^{-1/3}\sum_{l=1}^{l_L}[\Delta(b^{2-d})^l]^{2/3}K_l\,; \quad (8)$$

K_l is a positive random number of order unity. For $d>2$ the series (8) converges as $L\to\infty$; σ_0 is therefore well defined, from which the stability of the ferromagnetic ground state of the RFIM for small Δ can be inferred.

At finite T,

$$\sigma_{0b}(T,\Delta) \sim -b^{1-d}T\ln(Tb^2/J) - \tilde{C}^2\Delta(TJ)^{-1/2}b^{(3-d)/2}, \quad T \gg T_\Delta, \quad (9a)$$

$$\sigma_{0b}(T,\Delta) \sim E_{0b} - Tb^{1-d}\ln(\pi b^2/3J), \quad T \ll T_\Delta, \quad (9b)$$

where $\tilde{C}^2$ is a positive random number of order unity and $T_\Delta \equiv J^{-1/3}\Delta^{2/3}$.

If the random field is sufficiently weak that $T \gg T_\Delta$, then (9a) is appropriate; since $\Delta_l/T_l^{1/2} \sim (\Delta/T^{1/2})b^{l(3-d)/2}$, (8) comprises a series of terms of the form $b^{(3-d)l/2}$. The sum clearly diverges for $d<3$, suggesting $d_c=3$ at finite T. This conclusion is too hasty, however. Since $T_l/T_{\Delta_l} \sim (T/T_\Delta)b^{-(d+1)l/3}$, $T_l \ll T_{\Delta_l}$ for sufficiently large l, no matter how large T/T_Δ is initially. Indeed, $T_l/T_{\Delta_l} = 1$ when $l = l_c \equiv [3\ln(T/T_\Delta)]/(d+1)\ln b$. For $l > l_c$ (9a) no longer holds and (9b) must be substituted in (8), implying, for $d>2$, the convergence of the series and hence the existence of ferromagnetic order even at finite T. In other words, the variable T is sufficiently irrelevant that at large enough length scales the problem effectively reduces to $T=0$. That the $T > T_\Delta$ result (9a) suggests $d_c = 3$ is ultimately insignificant. Note that when $d<3$ the terms of (8) increase with l for $l<l_c$ [when (9a) holds] and decreases with l for $l>l_c$. The leading term of the series thus occurs when $l=l_c$ where both (9a) and (9b) are proportional to $\Delta^{2(d-1)/(d+1)}$. This is the result we quoted earlier.

The connection of these results with the roughness of the interface follows from (6) and $f = \sum_\lambda \varphi_\lambda q_\lambda$, which yield for the average width, w, of the interface at $T=0$

$$w^2 \sim \sum \varphi^2 q_0^2 \sim \Delta^{2/3}\sum_{l=1}^{l_L} b^{2(5-d)l/3}.$$

Here q_0 is the value of q which minimizes H_{0b} in (6). The results quoted earlier, $w^2 \sim \Delta^{2/3}L^{2(5-d)/3}$, $\Delta^{2/3}(\ln L)$, and $\Delta^{2/3}$ for $d<5$, $d=5$, and $d>5$, respectively, follow. These results, derived from our continuum interface model almost surely differ from those for the discrete lattice RFIM. The critical dimension d_R below which the Ising interface is always rough ($w=\infty$) is presumably lower than the value five predicted here. We argue this by analogy to the roughening transition in the pure Ising model, where continuum theories give $d_R=3$ whereas more careful treatments of lattice effects[6,12] show that actually $d_R=2$.

Since d_R for lattice RFIM is bounded above[13] by five (our continuum estimate), our $w(L)$ is presumably an upper bound for the true w of the lattice RFIM. Our conclusions that $w/L \to 0$ as $L \to \infty$ for $d=3$ and that $d_c=2$ should therefore be valid for discrete lattice RFIM's.

One simple test of the validity of our model can be performed at $T=0$ for $d=1$. The divergence of the ferromagnetic correlation length ξ as $\Delta \to 0$ at $T=0$ can be obtained from (4): $\Delta_l = \Delta e^{(2-d)l}$, where l is the logarithm of the length scale. Defining $l^*(\Delta)$ as the value of l at which $\Delta_l = 1$, one has $\xi \sim e^{l^*(\Delta)} \sim \Delta^{-(2-d)^{-1}}$ as $\Delta \to 0$. In one dimension, $\xi \sim \Delta^{-1}$, in agreement both with the domain argument[1] and exact calculations[14] on the one-dimensional RFIM; in two dimensions,[15] $\xi \sim e^{1/\Delta}$. Note that the recursion relations of Refs. 3 and 4 predict instead $\xi \sim \Delta^{-1/(3-d)}$ for $d<3$, or $\xi \sim \Delta^{-1/2}$ when $d=1$ and $\xi \sim \Delta^{-1}$ for $d=2$.

As mentioned above, experiments performed so far seem to show $d_c=3$ instead of 2. Furthermore, there have also been experiments[8] which showed $\xi \sim \Delta^{-1}$ for $d=2$. At present, we have no explanation of these experimental data.[16]

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Subsequent to the completion of this work we received a preprint from J. Villain who studied the commensurate-incommensurate transition in the presence of frozen impurities. Using arguments similar in spirit to ours he obtained identical results for w^2 and d_c. We are grateful to Dr. Villain for communicating this information to us prior to publication.

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[9]Continuum interface models such as ours and that of Refs. 3 and 4 typically overestimate the roughness, as discussed later in this paper.

[10]Though occasionally it will be convenient for us to argue in the replica formalism, all of our conclusions can be derived directly from (1) and an appropriate probability distribution for $h(\vec{x}, z)$.

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[13]Note that the continuum model of Ref. 3 gives $d_R=5$ but its discrete counterpart is presumed to have $d_R=4$.

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[15]To obtain $\xi \sim e^{1/\Delta}$, one assumes $\partial\Delta/\partial l \sim \Delta^2$.

[16]To have $\xi \sim \Delta^{-1}$ in both $d=1$ (as given by theory) and $d=2$ (as given by experiments of Ref. 8) would violate $\xi \sim \Delta^{-1/(d_c-d)}$, which ought to hold for *any* d_c. This is puzzling.