introduced form an interesting class of models. The "quantum" character of these classical models lies in their weights. Almost all vertex models studied so far have weights of the form $\omega_i = \exp(-\beta\epsilon_i)$, in which ϵ_i is an energy. Our weights are of a different nature: They are hyperbolic functions of the interaction energies. This has far-reaching consequences as can be seen by considering the uniform eight-vertex models with weights $\omega_{1H}-\omega_{8H}$. These models do not have the unphysical retention of correlations at infinite temperature as some of the conventional models with weights of the form $\exp(-\beta\omega_i)$ have.

We have shown that there exists an approximate mapping of the $S=\frac{1}{2}$ 2D XYZ model, which includes the Heisenberg and X-Y model, on a staggered eight-vertex model. This mapping results as the first approximation in the path-summation method of Suzuki, Miyashita, and Kuroda.[6] The X-Y version shows a phase transition without long-range order in the z magnetization, and without a divergence in the z susceptibility. This method introduces vertex weights of a different character from the classical ones. The study of these models is very interesting in itself.

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Irrational Decimations and Path Integrals for External Noise

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Irrational decimation schemes are constructed for functional integrals with the very interesting feature of preserving several distinct site Hamiltonians into the fixed-point limit. This method is applied to the quasiperiodicity transition to turbulence in order to compute the effects of external noise.

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In an important paper, Shraiman, Wayne, and Martin[1] determined the effect of noise on the period-doubling behavior of maps on an interval. In this paper we shall extend their methodology to determine the role of noise on the recently developed theory for the transition to chaos from quasiperiodic motion.[2,3] In order to perform our calculation, however, rather than using only existing renormalization ideas, we have constructed new decimation schemes that might prove useful in other contexts. Specifically, we construct *irrational* decimation schemes which at each level of renormalization produce *several* distinct Hamiltonians deployed along the lattice, approaching distinct fixed points. This is in contradistinction to the usual technique which produces the same Hamiltonian at each site, so that our technique accommodates dynamics whose critical behavior maintains multiple clusterings.

Let us recall Shraiman's adaption of decimation to the context of iterated maps. To a one-dimensional causal system we add external noise and so consider the stochastic process $x_{n+1} = f(x_n) + \xi_n$, where ξ_n is a noise sample drawn from a distribution with density ρ. Accordingly, $P_1(x_{n+1}|x_n) = \rho(x_{n+1}$

$-f(x_n))$ and the process is Markovian, so that the joint conditional generating function is

$$\hat{P}_n(k_n,\ldots,k_1|x_0,f)=\int\prod_1^n dx_i\exp(i\sum k_i x_i)\prod_0^{n-1}\rho(x_{i+1}-f(x_i)). \quad (1)$$

When ρ is written as e^H, (1) presents a path integral for a system whose Hamiltonian is the sum of identical, but lattice-index translated, copies of one-site Hamiltonians. We assume that ρ is Gaussian of width σ and attempt to use renormalization-group methodology to perform the functional integral. In order to understand the workings of this method, consider the $\sigma=0$ limit so that $\rho(x_{i+1}-f(x_i))=\delta(x_{i+1}-f(x_i))$. In fact, we shall set all k_i, $i\neq n$, in (1) to zero, and take $n=2^m$. The decimation is then set up as

$$\hat{P}_m(k|x_0,f)=\int\prod_1^{n/2}dx_{2i}\exp(ikx_n)\int\prod_1^{n/2}dx_{2i-1}\prod_0^{n/2-1}\delta(x_{2i+2}-f(x_{2i+1}))\delta(x_{2i+1}-f(x_{2i})). \quad (2)$$

The inner integral is trivial, so that

$$\hat{P}_m(k|x_0,f)=\int\prod_1^{n/2}dx_{2i}\exp(ikx_n)\prod_0^{n/2-1}\delta(x_{2i+2}-f^2(x_{2i})), \quad (3)$$

where $f^2(x)=f(f(x))$. Next the "spins" x_{2i} are rescaled according to $x_{2i}\equiv\bar{x}_i/\alpha$ (where α will be chosen to produce a fixed point), so that (3) becomes

$$\hat{P}_m(k|x_0,f)=\int\prod_1^{n/2}d\bar{x}_i\exp[i(k/\alpha)\bar{x}_{n/2}]\prod_0^{n/2-1}\delta(\bar{x}_{i+1}-(Tf)(\bar{x}_i)), \quad (4)$$

where T is the doubling transformation $(Tf)(x)=\alpha f^2(x/\alpha)$, and is the renormalization-group transformation on the Hamiltonian. By (1), (4) now establishes the scaling formula $\hat{P}_m(k|x_0,f)=\hat{P}_{m-1}(k/\alpha|\alpha x_0,Tf)$ which iterates to produce

$$\hat{P}_m(k|x_0,f)=\hat{P}_{m-r}(k/\alpha^r|\alpha^r x_0,T^r f). \quad (5)$$

Should f be "critical," then $Tf\to g$, where g is T's fixed point.

By the theory of period doubling,[4] when f is taken from a specified one-parameter family f_λ such that f_{λ_∞} is critical, there is a sequence of λ_n's convergent to λ_∞ with the property that

$$\lim_{r\to\infty}T^r f_{\lambda_r}=g_0,$$

where g_0 is a universal function. We use this fact as follows. By (5),

$$\hat{P}_m(k|x_0\lambda_r)=\hat{P}_{m-r}(k/\alpha^r|\alpha^r x_0,T^r f_{\lambda_r}). \quad (6)$$

Also, setting $m\to m+1$, $r\to r+1$ in (5), we have

$$\hat{P}_{m+1}(k|x_0,\lambda_{r+1})=\hat{P}_{m-r}(k/\alpha^{r+1}|\alpha^{r+1}x_0,T^{r+1}f_{\lambda_{r+1}}). \quad (7)$$

For r asymptotically large, we have upon comparing (6) and (7)

$$\hat{P}_m(k|x_0,\lambda_r)\sim\hat{P}_{m+1}(\alpha k|x_0/\alpha,\lambda_{r+1}). \quad (8)$$

From (8) we now extract a formula for the Lyapunov exponent, defined as

$$\Lambda(f)=\lim_{n\to\infty}\frac{1}{n}\ln\left\langle\frac{d}{dx_0}\langle f^n(x_0)\rangle_\rho\right\rangle_{x_0} \quad (9)$$

Noting (1), differentiate (8) on k, set $k=0$, differentiate on x_0, and average, so that

$$\Lambda(\lambda_r)\sim 2\Lambda(\lambda_{r+1})\quad\text{or}\quad\Lambda(\lambda_r)\sim k2^{-r}. \quad (10)$$

To extend (10) to $\sigma\neq 0$, replace in (2)

$$\delta(x_{2i+2}-f(x'))\delta(x'-f(x_{2i}))\to\frac{1}{2\pi\sigma^2}\exp(-\tfrac{1}{2}\sigma^{-2}\{[x_{2i+2}-f(x')]^2+[x'-f(x_{2i})]^2\}). \quad (11)$$

The result, for sufficiently small σ, must again be a normalized Gaussian,

$$\exp\{-[x_{2i+2}-f^2(x_{2i})]^2/2\sigma_1^{\,2}(x_{2i})\}/[2\pi\sigma_1^{\,2}(x_{2i})]^{1/2}. \tag{12}$$

Since $\sigma_1^{\,2}$ depends upon x_{2i}, assume σ^2 itself depends on x. Then to leading order in σ^2,

$$\tilde{\sigma}_{r+1}^{\;2}(x_{2i})=\sigma_r^{\,2}(f_r(x_{2i}))+[f_r'(f_r(x_{2i}))]_r^{\,2}\sigma^2(x_{2i}). \tag{13}$$

Upon spin rescaling of (12), (13) becomes

$$\sigma_{r+1}^{\;2}(x)=\alpha^2\{\sigma_r^{\,2}(f_r(x/\alpha))+[f_r'(f_r(x/\alpha))]^2\sigma_r^{\,2}(x/\alpha)\}. \tag{14}$$

Asymptotically, $f_r\to g$ and (14) is a fixed linear transformation, L, on $\sigma_r^{\,2}$. Denoting the largest eigenvalue and associated eigenfunction of L by β^2 and $\hat{\sigma}^2(x)$, we have $\sigma_r(x)\sim\beta^r\hat{\sigma}(x)$. We can now modify (6) to show the σ dependence:

$$\hat{P}_m(k|x_0,\lambda_r,\sigma)\sim\hat{P}_{m-r}(k/\alpha^r|\alpha^r x_0,T^r f_{\lambda_r},\beta^r\sigma) \tag{15}$$

which produces $\Lambda(\lambda_r,\sigma)\sim 2^n\Lambda(\lambda_{n+r},\sigma/\beta^n)$, or for some universal function F

$$\Lambda(\lambda_n,\sigma)\sim 2^{-n}F(\sigma\beta^n). \tag{16}$$

This completes our review of Shraiman's work.

The obvious weakness of alternating-site decimation is that only powers of f of the form 2^n can be constructed. Unless such iterates lie near a fixed point, such an organization of the integral is useless. We can extend the scope of usual decimations to produce p^n by performing $p-1$ contractions at a time. We symbolize this organization by a binary string with 1's representing sites to be integrated, so that

$$D_p\equiv\underbrace{011\cdots1}_{p}\ \underbrace{011\cdots1}_{p}\cdots \tag{17}$$

specifies the decimation that produces f^{p^n}. We shall call D_p a "decimation string" and the sequence of functions at successive lattice sites a "lattice string." Thus if we have the lattice string $L\equiv FFF\ldots$, D_p applied to it produces the new lattice string $L'=F'F'F'\ldots$, with $F'=F^p$. We next have to perform the spin rescaling $x_{pi}=\bar{x}_i/\alpha$. However, since L' is identical in form to L, this explicit rule is unnecessary: One need simply rescale the spins renamed serially from left to right.

A large class of interesting problems is specified by *irrational* values of p, in the sense that f^{F_n} approaches a fixed point where $F_n\sim\lambda^n$ with λ irrational. Precisely, with μ any quadratic irrational number (a number whose continued fraction expansion is periodic), there exists a map on the circle which possesses μ as its winding number (mean number of transits around the circle per iteration). According to very recent work,[2,3] one knows, at a critical parameter value $\omega_\infty(\mu)$,

$$\lim_{n\to\infty}\alpha^n f_{\omega_\infty(\mu)}{}^{F_n}(x/\alpha^n)=g_\mu(x), \tag{18}$$

with g_μ a universal function for each specified μ, where $F_{2n}/F_{2n+1}\to\mu$ as best rational approximants, and F_{2n}, $F_{2n+1}\sim\lambda^n$ with λ another quadratic irrational associated with μ. Also, there exists a sequence of ω_n's convergent to ω_∞ (at a given μ) with the property that

$$\lim_{n\to\infty}\alpha^n f_{\omega_n}{}^{F_n}(x/\alpha^n)=\hat{g}(x).$$

Given these facts one can hope that results like (16) should also exist. The obvious impediment is how to decimate at the irrational rate λ. Related to this, the analog to $2^n+2^n=2^{n+1}$ is $(F_{2n+2},F_{2n+1})=M\times(F_{2n},F_{2n-1})$ with M a 2×2 matrix with rows $(a\ b)$ and $(c\ d)$ formed as the product of p matrices, each of which has $m_{12}=m_{21}=1$, $m_{22}=0$, and $m_{11}=c_r$ when μ has the continued fraction with coefficients c_r of period p. (λ is the larger eigenvalue of M and $a+b\mu=\lambda$.) This implies that not only must the decimation, λ, be irrational, but moreover, that at each stage of decimation two *distinct* functions must reside on the lattice, so that contraction can reproduce the next higher level of the two functions. Also, the explicit rule of spin rescaling will not be available in closed form. However, these obstacles can be overcome.

First of all, in order to overcome the spin renaming problem, we demand that the lattice string after decimation be of the identical form, so that left-to-right serial renaming solves this difficulty. That two objects are present on the lattice simply means that the lattice string, L, is a binary string built out of the letters A and B. Upon applying D and renaming the new functions (A' and B') A and B, we demand that $L'=L$. That is, we must construct L as a *string fixed point* under the decimation D. If D is constructed to contract the δ functions according to the F_n recursion then all difficulties are resolved, and the functional integral exists.

We now construct L and D. First, note that

$$f^{F_{2n+2}} = \underbrace{f^{F_{2n}}\cdots f^{F_{2n}}}_{a}\ \underbrace{f^{F_{2n-1}}\cdots f^{F_{2n-1}}}_{b}, \tag{19}$$

where adjacent functions are functionally composed by the δ functions. Setting

$$f^{F_{2n}} \equiv A, \quad f^{F_{2n-1}} \equiv B, \quad f^{F_{2n+2}} \equiv A', \quad f^{F_{2n+1}} \equiv B',$$

gives

$$A' = A^a B^b, \quad B' = A^c B^d, \tag{20}$$

where A^a means a string of A's a long, and left-right ordering is preserved. The pieces of D that accomplish these contractions are

$$A^a B^b \to 01^{a+b-1} \text{ and } A^c B^d \to 01^{c+d-1} \tag{21}$$

with these pieces of D occupying the same sites as the A's and B's contracted. Finally we must determine the fixed string L. If we denote the transformation on L specified by (20) as S, L is also a fixed point of S^{-1}, so that the substitutions

$$A \to A^a B^b, \quad B \to A^c B^d \tag{22}$$

performed on L leave it invariant. Imagine an *a priori* chosen string L_0 whose first n letters agree with those of L. Since S contracts, SL_0 will agree on fewer letters. Conversely, $S^{-1}L_0$ agrees on more letters so that $L = \lim_{n\to\infty}(S^{-1})^n L_0$. That is, S^{-1} has L as its unique globally stable fixed point, so that starting with $L_0 = A$, successively applying (22) produces L. Finally, the substitution (21) determines the unique D that leaves L invariant, and the integral indeed exists.

We conclude this paper with an example, explicitly producing the analog of (18) for the quasiperiodicity transition at $\lambda = \mu^{-1} = \frac{1}{2}(5^{1/2}+1)$ (the "golden mean" with a continued fraction of all 1's.) In this case, $F_{2n+2} = F_{2n} + F_{2n-1}$, $F_{2n+1} = F_{2n}$, for which we have $AB \to A'$, $A \to B'$, so that we construct L from $A \to AB$, $B \to A$, resulting in

$$L = ABAABABAABAABABAABABA\cdots$$

and

$$D = 0100101001001010010 10\cdots .$$

By construction, D performs the integrations on all and only those δ functions which contain the lower number of iterates, and the ordering of a contraction is always

$$\int d\bar{x}\, \delta(x' - f_r(\bar{x}))\delta(\bar{x} - \alpha f_{r-1}(x/a)), \tag{23}$$

where

$$f_r(x) \equiv \alpha^r f^{F_r}(x/\alpha^r) \tag{24}$$

and the argument of the second δ function will be justified momentarily. The outcome of the integration (23) is $\delta(x' - f_r(\alpha f_{r-1}(x/\alpha)))$. We must next scale by α on x and x' so that we end up with

$$\delta(x' - \alpha f_r(\alpha f_{r-1}(x/\alpha^2))) = \delta(x' - f_{r+1}(x))$$

as follows from (24). However, at the *noncontracted* sites, we have $\delta(x' - f_r(x))$ which, when also rescaled, produces $\delta(x' - \alpha f_r(x/\alpha))$, justifying (23). Thus $A = \delta(x' - f_r(x))$, $B = \delta(x - \alpha f_{r-1}(x/\alpha))$, and at each level of decimation two distinct site Hamiltonians always appear, which at ω_∞ asymptotically approach $A = \delta(x' - g(x))$, $B = (x' - \alpha g(x/\alpha))$.

To produce the scaling formulas, rather than $n = 2^m$ in (2) and what follows it, we now take $n = F_{2m}$, which modifies (10) to

$$\Lambda(\omega_r) \sim (F_{2m+2}/F_{2m})\Lambda(\omega_{r+1}) \sim \lambda\Lambda(\omega_{r+1}),$$

so that the causal Lyapunov formula is $\Lambda(\omega_r) \sim k\lambda^{-r}$.

Finally, we add noise by replacing the δ functions by Gaussians of width σ. Analogously to (23), the exponent is now

$$[x' - f_r(\bar{x})]^2/\sigma_r^2(\bar{x}) + [x - \sigma f_{r-1}(x/\alpha)]^2/\alpha^2\sigma_{r-1}^2(x/\alpha),$$

which results in the modification to (14) which now reads

$$\sigma_{r+1}^2(x) = \alpha^2\{\sigma_r^2(\alpha f_{r-1}(x/\alpha^2)) + \alpha^2[f_r'(\alpha f_{r-1}(x/\alpha^2))]^2\sigma_{r-1}^2(x/\alpha^2)\}. \tag{25}$$

Using the known numerical form for $g(x)$ in this case,[3] we obtain the leading eigenvalue $\beta_\mu = 2.306\ldots$, and the main scaling result (16) becomes

$$\Lambda(\omega_n, \sigma) \sim \lambda^{-n} F_\mu(\sigma\beta_\mu{}^n). \tag{26}$$

Equation (26) has the simple consequence that if σ_n is defined as the strength of noise to destabilize the cycle of length F_n at ω_n [i.e., $\Lambda(\omega_n, \sigma_n) = 0$], then $\sigma_n \sim \beta_\mu{}^{-n}$ which we have numerically verified to two significant figures. Obviously, our method produces (26) for any quadratic irrational μ, although formulas like (25) become somewhat more complicated and require new numerics to produce the corresponding β_μ.

We have presented a method for arbitrary lattice decimation. The scheme of (22) can obviously be extended to lattice strings with any number of distinct Hamiltonians, with higher-dimensional F_n schemes. It is conceivable that dynamics exist that can utilize arbitrarily large numbers of distinct site Hamiltonians, which in the infinite limit might determine unusual critical behaviors. Finally, the lattice strings we produce scale into themselves through the decimation S, so that they are of a fractal structure, and of intrinsic interest.

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