Size Scaling for Infinitely Coordinated Systems

R. Botet, R. Jullien, and P. Pfeuty
Laboratoire de Physique des Solides, Université de Paris-Sud, Centre d'Orsay, F-91405 Orsay, France
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The finite-size scaling of Fisher and Barber is extended to infinitely coordinated systems. Near T_c and for a large number of elements N, a critical quantity A behaves as $|T-T_c|^a f(N/N_c)$ with $N_c \sim |T-T_c|^{-\nu^*}$. An argument gives $\nu^* = \nu_{MF} d_c$, where ν_{MF} is the mean-field coherence-length exponent and d_c the upper critical dimensionality of the corresponding short-range system. This is checked on spin systems at $T \neq 0$ and on the Ising-XY quantum spin system in a transverse field at $T=0$ for which calculations are reported.

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An infinitely coordinated system is a system of N elements each of which is coupled to all others with a strength which does not depend on the position and nature of the interacting elements. Such systems, which are particularly simple, have been widely studied. In some cases their thermodynamical properties can be derived analytically,[1,2] so that they provide good examples to illustrate the general theory of phase transitions. In the thermodynamic limit ($N \to \infty$) such systems may present a true phase transition. This transition, which can be often studied exactly, is also correctly described by a mean-field approach which becomes adequate because of the long-range nature of the interactions. This explains why they have often been introduced as a first "mean-field like" model of real d-dimensional short-range systems.[3] However, when N is large but not infinite no true transition exists; but for systems which present for $N \to \infty$ a second-order phase transition there is still a critical scaling of the thermodynamical quantities with N, which depends on the nature of the system: for instance, Kittel and Shore[1] found analytically that at the critical temperature the magnetization goes to zero as $N^{-1/4}$ for Heisenberg as well as for Ising infinitely coordinated spin-$\frac{1}{2}$ systems. They were already surprised by so slow a decay of the critical magnetization which did not come out very simply from their calculations.

In this Letter we would like to extend the finite-size scaling hypothesis of Fisher and Barber[4] to these systems. As a result of the infinite-range nature of the interactions the concepts of "dimensionality" as well as "length" have lost their meaning. The coherence length ξ must be replaced by a more general quantity, independent of the dimensionality, a "coherence number" N_c which is supposed to diverge at the transition, in the infinite system, as

$$N_c \underset{N=\infty}{\sim} |T-T_c|^{-\nu^*}, \tag{1}$$

where ν^* is a given exponent depending on the considered system but independent of dimensionality. Consider a given critical quantity A which, in the infinite system, behaves near T_c as

$$A \underset{N=\infty}{\sim} (T-T_c)^a. \tag{2}$$

a is the mean-field exponent for the singularity of A at the critical point. The scaling hypothesis is simply extended by assuming the existence of a regular function $F_a(x)$, such that, near T_c and for large N, A can be written as

$$A \sim |T-T_c|^a F_a(N/N_c). \tag{3}$$

We suppose that $F_a(x) \to$ const when $x \to \infty$ such that (2) is recovered. When $x \to 0$, we suppose that $F_a(x) \sim x^{\omega_a}$, with $\omega_a = -a/\nu^*$ in order that A would be regular at T_c for large but finite N. The direct consequence is that at $T=T_c$ one has

$$A \underset{T=T_c}{\sim} N^{\omega_a} \sim N^{-a/\nu^*}. \tag{4}$$

It is thus sufficient to know the exponent ν^*, as well as the mean-field exponents of the system, to know how the thermodynamical quantities scale with size at criticality.

ν^* can be a simple conjecture based on two assumptions. The corresponding short-range system with dimensionality $d=d_c$ where d_c is the upper critical dimensionality behaves with mean-field behavior. The first assumption is that the scaling theory, as developed by Fisher and Barber,[4] would apply to this short-range system, with a scaling form similar to Eq. (3) with $N_c \sim \xi^{d_c}$ where ξ is the coherence length diverging at T_c with the mean-field exponent ν_{MF}, $\xi \sim (T$

$-T_c)^{-\nu_{\rm MF}}$. The second assumption is that both systems, the infinitely coordinated system with N elements and the corresponding short-range system at $d=d_c$ with size $L\sim N^{1/d_c}$, would have the same scaling exponents. It follows that

$$\nu^* = \nu_{\rm MF}\, d_c\,. \qquad (5)$$

As a direct consequence the exponents of the large-N behavior of all quantities of the infinitely coordinated system at criticality are related to d_c. For example, for the magnetization, one has

$$m \underset{T=T_c}{\sim} N^{\omega_m} \text{ with } \omega_m = -\beta_{\rm MF}/\nu_{\rm MF}\, d_c\,. \qquad (6)$$

In several systems, such as the $O(n)$ vector model, $\nu_{\rm MF} = \beta_{\rm MF} = \frac{1}{2}$ so that the formula reduces simply to $\omega_m = -1/d_c$.

Before discussing some examples, we must be more precise concerning the above conjecture. It has been shown that the first assumption is not true for the short-range spherical model for which the scaling form is not so simple as Eq. (3).[5] At $d=d_c$, the fluctuations play some role in the short-range system. Even if they do not change the mean-field exponents, they introduce some logarithmic terms in the usual scaling [L/ξ is replaced by $(L/\xi)(\ln L)^\alpha$]. However, it could be expected that the fluctuations vanish completely in the uniform infinitely coordinated limit so that the logarithmic term would disappear. At this stage, this is only an assumption and our conjecture must be checked on examples before being trusted.

The conjecture is verified for Ising as well as for Heisenberg spin systems as shown by the result of the calculation by Kittel and Shore.[1] For the corresponding short-range system the transition is driven by thermal fluctuations, so that the system behaves "classically." The upper critical dimensionality is thus $d_c=4$ leading to $m\sim N^{-1/4}$ at $T=T_c$ for the corresponding infinitely coordinated models (without any logarithmic term). This is already a good check, but to show the generality of these considerations, we have performed some numerical calculations on other systems for which the upper critical dimensionality is different from 4.

Let us report now the results of calculations done on the infinitely coordinated version of the quantum Ising-XY model in a transverse field at $T=0$. Such a system has received a peculiar interest in the short-range case and has been exactly solved in one dimension.[6] It is well known that in the anisotropic case and in d dimensions such a system develops at $T=0$ a phase transition in field equivalent to the transition in temperature of the $(d+1)$-dimensional classical Ising system.[7] The gap G between the ground state and the first excited state is proportional to the inverse of the coherence length in the extra dimensionality for the $(d+1)$-dimensional equivalent. It results that the dynamical exponent z (which tells how the gap scales with length L at the critical field: $G\sim L^{-z}$) is exactly equal to 1. Also, from this equivalence, it follows that the upper critical dimensionality is reduced by one so that $d_c=3$ in the anisotropic case. The fully symmetric XY model is very peculiar since this equivalence does not hold. The exponent z is equal to 2 instead and it has been suggested that the system would be equivalent to a $(d+2)$-dimensional classical model so that the upper critical dimensionality would be $d_c=2$ in the symmetric case.[8]

We have considered the following spin-S quantum Hamiltonian:

$$H = -\frac{1}{NS}\sum_{i<j}(S_i^x S_j^x + \gamma S_i^y S_j^y) - h\sum_i S_i^z \qquad (7)$$

for $0\leq\gamma\leq 1$. $\gamma\neq 1$ corresponds to the anisotropic case while $\gamma=1$ corresponds to the isotropic case. The $1/N$ prefactor is essential to insure the convergence of the ground-state energy per spin in the thermodynamic limit where the system develops a mean-field transition in the ground state for $h=h_c=1$. Equation (7) can be transformed into

$$H = -(1/2NS)\{J^{x2} + \gamma J^{y2} - K^x - \gamma K^y\} - hJ^z \qquad (8)$$

with

$$J^\alpha = \sum_i S_i^\alpha, \quad K^\alpha = \sum_i (S_i^\alpha)^2\,. \qquad (9)$$

This form permits a better use of the symmetries. In the thermodynamic limit the K terms become irrelevent and it appears that the lowest states belong to the maximum eigenvalue $J=NS$ of the total spin so that the study of (8) for $N=\infty$ can be replaced by the study of

$$H = -(1/2J)(J^{x2} + \gamma J^{y2}) - hJ^z\,. \qquad (10)$$

This peculiar simple form which involves a unique, but very large, spin of size $J=NS$ is well known as the Lipkin model in nuclear physics.[9] This form (10) can be used in the case $S=\frac{1}{2}$ *even for finite N* since the K's are then simple constants. However, for $S>\frac{1}{2}$ we must consider the general form (8).

We have defined the magnetization m in the ground state as being given by

$$m^2 = (NS)^{-2} \langle 0 | J^{x2} | 0 \rangle. \quad (11)$$

We have calculated m as well as the gap G between the ground state and the first excited state for finite sizes up to $N = 150$, 70, 8, respectively, for $S = \frac{1}{2}$, 1, $\frac{3}{2}$ in the general case $\gamma \neq 1$. The isotropic case $\gamma = 1$ can be entirely treated analytically for $S = \frac{1}{2}$.[10]

The mean-field and random-phase approximation expressions[11] of m and G are given by

$$m_\infty = (1 - h^2)^{1/2}, \quad G_\infty = 0 \text{ for } h < 1, \qquad m_\infty = 0, \quad G_\infty = [(h - \gamma)(h - 1)]^{1/2} \text{ for } h > 1, \quad (12)$$

so that their corresponding critical exponents usually denoted as β and s are

$$\beta = \tfrac{1}{2} \text{ for all } \gamma; \qquad s = \tfrac{1}{2} \text{ for } \gamma \neq 1, \quad s = 1 \text{ for } \gamma = 1. \quad (13)$$

Typical finite-N results are shown in Fig. 1 in the $S = \frac{1}{2}$ Ising case ($\gamma = 0$) where $m(h)$ and $G(h)$ have been plotted for $N = 20$, 60, 100, and compared with expressions (12) (dashed curves). We have determined the asymptotic large-N behavior numerically and we have found for all $\gamma \neq 1$

$$m - m_\infty \sim N^{-1}, \quad G \sim \exp(-aN) \text{ for } h < 1; \qquad m \sim N^{-1/3}, \quad G \sim N^{-1/3} \text{ for } h = 1; \qquad m \sim N^{-1/2}, \quad G - G_\infty \sim N^{-1} \text{ for } h > 1. \quad (14)$$

Moreover we have precisely verified that both m and G verify the scaling form (3) [i.e., we have been able to determine $F_m(x)$ and $F_G(x)$] with

$$\omega_m = \omega_G = -\tfrac{1}{3}, \quad \nu^* = \tfrac{3}{2} \text{ for } \gamma \neq 1. \quad (15)$$

In particular we did not detect any logarithmic term. The exponent $\frac{1}{3}$ for G at critical field seems to be independent of S as shown by further calculations for $S = 1$, $\frac{3}{2}$ which will be reported elsewhere.[12]

In the fully isotropic case $\gamma = 1$, a simple analytical treatment[10] gives

$$G - G_\infty \sim N^{-1} \text{ everywhere}, \qquad m - m_\infty \sim N^{-1} \text{ for } h < 1, \qquad m \sim N^{-1/2} \text{ for } h \geq 1. \quad (16)$$

The scaling forms (3) are also verified with

$$\omega_m = -\tfrac{1}{2}, \quad \omega_G = -1, \quad \nu^* = 1 \text{ for } \gamma = 1. \quad (17)$$

When comparing our results for m with those obtained by Kittel and Shore[1] for the transitions of spin systems in temperature we observe that, for $\gamma \neq 1$, we recover exactly the same scaling properties below, as well as above, the transition. This shows that the ordered, as well as the disordered, phase is the same in both cases (is described by the same trivial fixed point). However, we find different exponents at the critical point. The exponent $\omega_m = -\frac{1}{3}$ is characteristic of quantum fluctuations while the exponent $\omega_m = -\frac{1}{4}$ of Kittel and Shore is characteristic of classical thermal fluctuations. For $\gamma \neq 1$, we expect the same kind of quantum-classical crossover phenomenon as in the short-range case[13] but with different exponents. In the (h, T) plane, the ordered and disordered phases are separated by a critical line $h_c(T)$ starting at $h_c = 1$, $T = 0$ and ending at $h = 0$, $T = T_c$. At all points of this line the asymptotical decay of the magnetization is classical, i.e., $m \sim N^{-1/4}$, except just at $h_c = 1$, $T = 0$ where it is quantum, i.e., $m \sim N^{-1/3}$. At a very small temperature on the critical line we can define a crossover number N^* such that $G \sim kT \sim (N^*)^{-\omega_G}$. For $N < N^*$ ($N^* \sim kT^{-3}$) the system seems to behave quantally, i.e., $m \sim N^{-1/3}$, while for $N > N^*$ it finally behaves classically, i.e., $m \sim N^{-1/4}$.

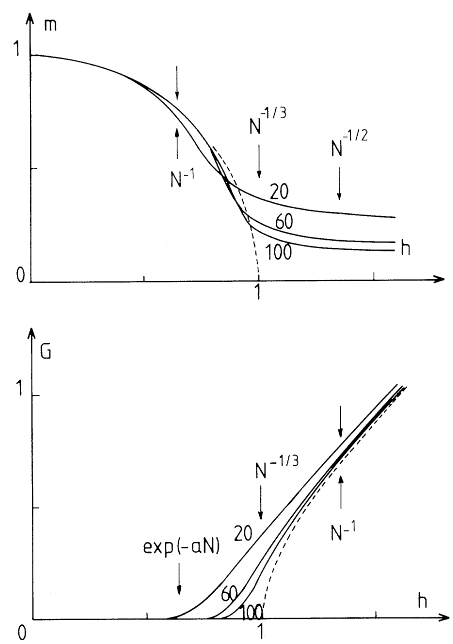

FIG. 1. Finite-size results ($N = 20, 60, 100$) for the magnetization and the gap of the infinite ranged Ising model in a transverse field.

The main results for ν^* obtained for two quantum systems [Eqs. (15) and (17)] for which d_c is equal respectively to 3 and to 2, together with the old results for classical spin systems[1] for which $d_c = 4$, fully confirm the consequences of the main assumption presented in this Letter and given in Eq. (5) where $\nu_{\rm MF}$ is equal to $\frac{1}{2}$ for the systems studied here.

In this Letter we have given a simple argument relating the critical scaling with N in infinitely coordinated systems with the upper critical dimensionality of the corresponding short-range system. This argument has been checked here in three different cases. More details and further checks will be reported elsewhere,[12] in particular in the case of a quantum infinitely coordinated version of the Ising model in a complex field, the so-called Lee-Yang problem (for which[14] $d_c = 6$). It would be interesting also to extend the arguments to disordered systems such as spin-glass[3] or systems with a random field.[15]

Another consequence of formula (6) is that it provides a new way to determine the upper critical dimensionality d_c of a given system from the size scaling at criticality of the corresponding infinitely coordinated model.

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[10]For $\gamma = 1$ the spectrum of (10) is $E(J,M) = -(J+1)/2 + M^2/(2J) - hM$ where $M = J, J-1, \ldots, -J$ is the eigenvalue of J^z. For $h > 1$ the lowest eigenvalues correspond to $M = J, M = J-1$ so that $G = h - 1 + 1/(2J)$ and $m \sim J^{-1/2}$. For $h < 1$ the M value changes discontinuously with h; however, for J large one has $M \sim hJ$ so that $G \sim 1/(2J)$ and $m = m_\infty + O(1/J)$.

[11]The mean-field expression of m is straightforwardly obtained by considering $\vec{J}$ as a classical spin, $J^x = J \times \sin\theta\cos\varphi$, $J^y = J\sin\theta\sin\varphi$, $J^z = J\cos\theta$ and by minimizing the energy with respect to θ and φ. The expression for G is obtained by deriving the equations of motion of J^x and J^y. The random-phase approximation consists in replacing J^z by its mean-field value in these expressions. The gap (in units $\hbar$) is identified with the frequency obtained.

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